Farsighted Stability in Patent Licensing: An Abstract Game Approach*

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Abstract

This paper analyzes the negotiations made by an external patent holder and potential licensee firms in a new model of patent licensing, assuming that they are all farsighted, and characterizes the symmetric farsighted stable sets. Given a net profit of each licensee firm, a set of outcomes is a symmetric farsighted stable set if and only if, at any outcome in the set, each licensee firm receives the net profit and the number of licensee firms maximizes the patent holder's profit provided that licensee firms obtain the net profits. We also show the close relationship between the symmetric farsighted stable sets and the relative interior of the core. Further, we confirm that our result is strong and robust by applying the notions of absolute maximality (Ray and Vohra, 2019a) and history dependent strong rationality (Dutta and Vartiainen, 2017) to our model, respectively; the symmetric farsighted stable sets are the absolutely maximal farsighted stable sets and with the history dependent strongly rational expectation farsighted stable set.

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1 Introduction

This paper analyzes the negotiations on prices of information about patented technologies under a situation where the seller of the information (patent holder) has no production facility, the buyers (licensee firms) enjoy an advantage, but the non-buyers (non-licensee firms) suffer from a disadvantage in the market competition. License agreements themselves that result from negotiations have been analyzed by researchers from the viewpoint of cooperative games.[a] We argue the process of license negotiations itself as well as a set of agreements.

A cooperative game approach to patent licensing was initiated by Tauman and Watanabe (2007), where the grand coalition was supposed to be formed. Another cooperative game approach was developed by Watanabe and Muto (2008), where a coalition structure was selected by the patent holder. Suppose that there are an arbitrarily finite number of firms competing in the market. In the first stage, a patent holder invites a group of firms to license negotiations. In the second stage, the patent holder and the invited firms negotiate on the fees for the patented technology, assuming that the patented technology is licensed to all firms invited to this negotiation. In the third stage, the licensee firms and non-licensee firms compete in a market, provided that the results of the first and second stages are commonly known to all the firms.[b]

In the model of Watanabe and Muto (2008), a patent holder and firms make their decisions in the first and second stages with foreseen (re)actions of firms in the market competition. Then, it is also natural to introduce a situation where a patent holder and firms also foresee reactions of others in license negotiations, as Diamantoudi (2005) emphasized for a cartel formation. In order to formulate license negotiations made by such a patent holder and firms who are all farsighted, we combine the first and second stages and formulate license negotiations with an abstract game due to Chwe (1994) in

[a]Patent licensing has been analyzed also with non-cooperative game approach since seminal papers by Kamien and Tauman (1984, 1986). See, e.g., Sen and Tauman (2007), Fan et al. (2016), and references therein for this strand of research.

[b]Many bargaining solutions have been studied in this model; the core and bargaining set by Watanabe and Muto (2008), the Shapley-Aumann-Drèze value by Kishimoto et al. (2011), the kernel and nucleolus by Kishimoto and Watanabe (2017), and the von Neumann-Morgenstern stable sets by Hirai and Watanabe (2018). Kishimoto (2013) extended this model to a game with non-transferable utility.

the following way. A proposal of licensing the patented technology and its fee to some firms can be altered with another proposal to other firms, or it can be canceled. Other proposals or cancellation may follow after a proposal and cancellation.

Farsighted stable set is an appropriate solution concept for such a negotiation process mentioned above, which was introduced by Harsanyi (1974) for coalitional games and reformulated by Chwe (1994) for a class of abstract games.[c] We show as a main result that, under a remarkably mild condition, a set of symmetric outcomes is a symmetric farsighted stable set if and only if the set satisfies the following condition for a given sufficiently small positive net profit for licensee firms. For any outcome in the set, (i) licensee firms uniformly enjoy a given net profit, and (ii) the number of licensee firms maximizes a patent holder's profit, provided that the payoffs for licensee firms are determined as in (i). By the finiteness of firms, this result also implies the existence of a symmetric farsighted stable set.

Our new model of licensing negotiations and a farsighted stable set supports a remarkably different manner of profit sharing from the previously reviewed literature. In previous studies, a patent holder determines the number of licensee firms to maximize own profit ex ante with the foreseen result in the subsequent license negotiations and market competition. A farsighted stable set in our model offers another way to determine a patent holder's profit. A net profit of licensee firms is determined first. Then, the number of licensee firms is determined as if a patent holder maximizes its own profit ex post provided that licensee firms obtain the given net profit, which might be interpreted as an accepted standard of behavior or established order of society.[d]

From this characterization of symmetric farsighted stable sets, it will be easily confirmed that a singleton and symmetric farsighted stable set exists if and only if licensing all firms is the unique maximizer of the patent holder's profit for a given net profit of licensee firms. It will also be shown that this condition is necessary and sufficient for the nonemptiness of the relative interior of the core. There are some literature that showed a close relationship between the core and farsighted stable sets that yield a single payoff vector (Mauleon, et al., 2011; Ray and Vohra, 2015a; Chander, 2015; Hirai, 2018). Unlike

[c] Farsighted stable set is a modified version of the von Neumann-Morgenstern stable set for farsighted players. Following Harsanyi (1974) and Chwe (1994), there is a large literature on solution concepts that are based on farsighted players. Ray and Vohra (2014) give an insightful survey on that topic.

[d] von Neumann and Morgenstern (1944) considered social systems stylized in real practices as outcomes of negotiations made by people who are faced with non-cooperative situations, and proposed a solution which describes agreements people reach eventually in those negotiations, which can be interpreted as an established order of society or accepted standard of behavior.

these literature, there is a singleton farsighted stable set that is not included in the core in our model. The equivalence between singleton farsighted stable sets and the relative interior of the core is obtained if we restrict our attention to symmetric farsighted stable sets. This result implies that singleton and symmetric farsighted stable sets have strong stability.

Further, we apply the notions of history dependent strong rationality developed by Dutta and Vartiainen (2017) and absolute maximality introduced by Ray and Vohra (2019a) to our model where the patented technology causes non-licensee firms a negative externality through the market competition with the licensee firms. Dutta and Vohra (2017) pointed out that a coalition may intervene paths (chain reaction of proposals and counter-proposals) constituting an indirect dominance relation to make its members even better, where the intervening coalition can possibly be different from those appeared in the paths. Namely, indirect domination itself "does not require coalitions to choose their best moves and rules out possible unwelcome intervention by other coalitions" (Ray and Vohra, 2019a, Section 1). In order to guarantee that an indirect dominance relation constructed from a negotiation process is robust to such an intervention by a coalition, Dutta and Vartiainen (2017) applied history dependent strong rationality to a negotiation process, while Ray and Vohra (2019a) introduced absolute maximality which is "something like sequential rationality" into individual decision making in a negotiation process.[e] We show that the symmetric farsighted stable sets coincide with the absolutely maximal farsighted stable sets as well as the history dependent strongly rational expectation farsighted stable set in our model, which implies that our main result noted above is robust.

The remainder of the paper is organized as follows. In Section 2, we formulate a model of patent licensing negotiations as an abstract game. In Section 3, we define a (symmetric) farsighted stable set of the game. The main results are stated and proved in Section 4. The relationship with the core is mentioned in the latter part of that section. At the end of this section, we briefly refer to the difference of our solution from

[e]Dutta and Vohra (2017) considered a negotiation process (a transition from a state to another as well as the coalition which is supposed to effect the move) as an expectation function and proposed a farsighted stable set based on an expectation function that depends only on the current outcome, and they also considered a weaker notion of the maximality. (The idea of expectation function originally came from Jordan (2006).) Dutta and Vartiainen (2017) and Ray and Vohra (2019a) refined the seminal ideas that Dutta and Vohra (2017) provided, respectively. Bloch and van den Nouweland (2017) allowed players to have heterogeneous expectations over the dominance paths and considered expectation functions satisfying path-persistence and consistency.

the classical solution concepts defined by von Neumann and Morgenstern (1944) with a simple example. In Section 5, we examine history dependent strong rationality and absolute maximality for the farsighted stable sets derived in Section 4. We will describe those notions in detail there. In Section 6, we close this paper with some remarks.

2 A model of patent licensing

A patented technology is held by an agent, who is denoted by 0. Assume that the patent is perfectly protected; no firm can use the patented technology without the patent holder's permission. The patent holder has no production facility, and thus it obtains nothing from the patented technology unless it sells the license. This agent is called an external patent holder.[f] Let $N = \{1, 2, \ldots, n\}$, where $2 \leq n < \infty$, be the set of firms that have an identical production technology before a patented technology is licensed. Firms are symmetric in this sense.[g] These firms compete in an oligopoly market. Thus, $\{0\} \cup N$ is the set of players. A nonempty subset of $\{0\} \cup N$ is called a coalition. Throughout this paper, we denote coalitions by capital letters and their cardinalities by the corresponding lower cases. For example, the cardinalities of coalitions $S, T, \bar{S}, S'$, and Q^h are denoted by $s, t, \bar{s}, s'$, and q^h, respectively.

This game consists of two stages. In the first stage, players negotiate on (i) which firms are licensed and (ii) how much licensee firms have to pay to the patent holder. We call this stage the negotiation stage. In the second stage, firms compete with each other in a market, provided that the outcome of the negotiation stage is common knowledge to all the firms. We call this stage the market competition stage. Firms are prohibited from forming any cartels to coordinate their production levels and market behaviors, because they cannot make binding agreements on such cartels. This is the assumption for the comparison of the bargaining outcomes with the non-cooperative outcomes in the traditional literature. A coalition is thus formed only for license negotiations.[h]

Since firms are identical before licensed, the payoff of a firm resulting from the market depends only on whether it is licensed and the number of firms licensed in the negotiation

[f]Research laboratories and engineering departments at universities are typical examples of such agents, because they do not have any production facilities.

[g]We keep this assumption in the traditional literature of patent licensing intact.

[h]In this sense, the model is different from coalition formation games. Among those games, e.g., Konishi and Ray (2003) analyzed a process of coalition formation from a viewpoint of an equilibrium concept closely related to farsighted stability.

stage. When $s(=0,...,n)$ firms are licensed, let $W(s)$ denote the payoff of a licensee firm and $L(s)$ denote the payoff of a non-licensee firm at the market competition stage. For notational convenience, $W(0)$ and $L(n)$ are assumed 0. Throughout the paper, the following assumption is imposed.

Assumption 1 *(i)* $W(s) > L(0)$ *for all* $s = 1,...,n$*; (ii)* $L(0) > L(s) \geq 0$ *for all* $s = 1,...,n-1$.

This assumption implies that the patented technology is advantageous for licensee firms and disadvantageous for non-licensee firms when it is licensed.

We aim to find stable outcomes at the negotiation stage, where players negotiate with the foreseen payoffs (profits) that will be obtained at the subsequent market competition stage. Players negotiate to agree on a contract that determines which firms are licensed and the fees licensee firms pay. We allow asymmetric fees; The licensee firms may pay different fees to the patent holder. On the other hand, we assume that non-licensee firms pay nothing. We assume that any contract is multilateral in the sense that a contract is broken up unless all of the patent holder and licensee firms agree on it.

Contracts represent outcomes of the negotiation stage. An outcome of the negotiation stage is a pair of a set of licensee firms and a payoff allocation where there are transfers between a patent holder and licensee firms. Formally, for a given $S \subseteq N$, let

$$X^S = \left\{ x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} \,\middle|\, x_0 + \sum_{i \in S} x_i = sW(s),\ x_j = L(s) \text{ for all } j \in N \setminus S \right\}$$

denote the set of feasible payoff allocations when firms in S are licensed. Note that $X^{\emptyset} = \{(0, L(0), ..., L(0))\}$. We denote $x^{\emptyset} = (0, L(0), ..., L(0))$. Let

$$X = \bigcup_{S \subseteq N} \left(\{S\} \times X^S \right)$$

denote the set of outcomes. Let $\bar{X} = \{(S, x) \in X | S \subseteq N, x_i = x_j, \forall i, j \in S\}$ be the set of symmetric outcomes. By the symmetry of firms, licensee firms are paying a uniform fee at an outcome if and only if the outcome is symmetric.

We turn to the definition of an effectiveness relation on X that describes rules of the negotiation. An effectiveness relation determines coalitions that are able to induce an outcome from an outcome. We denote $(S, x) \rightarrow_T (S', x')$ when a coalition T can induce $(S', x') \in X$ from $(S, x) \in X$. Since we are assuming that contracts in the negotiation are multilateral, we impose the following assumptions on effectiveness relation.

Assumption 2 *(i) For any* $(S,x) \in X$, $(S,x) \to_T (\emptyset, x^{\emptyset})$ *if and only if* $\emptyset \neq T \subseteq \{0\} \cup S$*; (ii) For any* $(S,x),(S',x') \in X$ *with* $S' \neq \emptyset$, $(S,x) \to_T (S',x')$ *if and only if* $T = \{0\} \cup S'$.

Assumption 2 requires that agreements from all members are necessary (i) to maintain a contract or (ii) to make a new contract. In (i), when a coalition $T \subseteq S$ deviates from (S,x), the patent holder and the residual firms $\{0\} \cup (S \setminus T)$ are not allowed to keep the licensing contract between only them.[i] This contrasts with the effectivity considered in some literature of farsighted stability, for example, the hedonic games considered by Diamantoudi and Xue (2003) and general partition function games considered by Chander (2015), among others. Rather, we inherit the nature of the negotiation process in the cooperative patent licensing game since Watanabe and Muto (2008). In (ii), it is legitimate that T can break up (S,x) before making (S',x') because $0 \in T = \{0\} \cup S'$ in this case, even if $S \neq \emptyset$. Note that (ii) reduces to a redistribution when $S = S' \neq \emptyset$, *i.e.* for any nonempty $S \subseteq \{0\} \cup N$ and $x, x' \in X^S$, $(S,x) \to_{\{0\}\cup S} (S,x')$.

In our effectiveness relation, a coalition may affect the payoffs of players outside the coalition when it induces a new outcome. For example, when $(\emptyset, x^{\emptyset}) \to_{\{0\}\cup S} (S,x)$ takes place for some $S \neq \emptyset, N$, the players in $N \setminus S$ are made worse off. Such an effect is due to a negative externality of the patented technology to non-licensee firms, not a violation of (coalitional) sovereignty pointed out by Ray and Vohra (2015a). Note also that our model is essentially an abstract game due to Chwe (1994).[j]

3 Farsighted stable set

In the negotiation stage, we assume that the players make their decisions with the foreseen payoffs obtained at the subsequent market competition stage. In this sense, it is implicitly assumed that the players are farsighted in the model. Therefore, it seems consistent to consider a stability notion for farsighted players as a solution concept. We employ the farsighted stable set that satisfies the stability notions *à la* von Neumann and Morgenstern (1944), where those stability notions are defined according to indirect dominance relations.

[i] In the Appendix, we examine how our results vary under an effectiveness relation where the patent holder and residual firms can keep the licensing contract when some licensee firms deviate from the contract.

[j] An abstract game is originally defined as a quadruple of a set of players, a set of outcomes, a profile of each player's preference over the set of outcomes, and a profile of effectiveness relation. In our model, players' preferences are not specified because outcomes directly represent their payoffs.

We first introduce the definition of an indirect dominance relation.

Definition 1 *Let $(S,x),(S',x') \in X$. We say that (S',x') indirectly dominates (S,x), which is denoted by $(S',x') \succ (S,x)$, if and only if there exist a sequence of outcomes $(S^0,x^0),...,(S^m,x^m)$ and a sequence of coalitions $T^1,...,T^m$ such that $(S^0,x^0)=(S,x)$, $(S^m,x^m)=(S',x')$, and for all $h=1,...,m$,*

- $(S^{h-1},x^{h-1}) \rightarrow_{T^h} (S^h,x^h)$;
- $x'_i > x_i^{h-1}$ *for all* $i \in T^h$.

For simplicity, we sometimes denote the sequences of outcomes and coalitions yielding an indirect dominance relation $(S',x') \succ (S,x)$ as the following paths.

$$(S,x)=(S^0,x^0) \rightarrow_{T^1} (S^1,x^1) \rightarrow_{T^2} \cdots \rightarrow_{T^m} (S^m,x^m)=(S',x').$$

Then, a farsighted stable set is defined as follows. We also define a symmetric farsighted stable set.

Definition 2
- *We say that $K \subseteq X$ is a farsighted stable set if and only if the following two stabilities are satisfied.*

  Internal stability: *for any $(S,x),(S',x') \in K$, $(S',x') \succ (S,x)$ does not hold.*

  External stability: *for any $(S,x) \in X \setminus K$, there exists some $(S',x') \in K$ such that $(S',x') \succ (S,x)$.*

- *We say that $K \subseteq X$ is a symmetric farsighted stable set if and only if K is a farsighted stable set and $K \subseteq \bar{X}$.*

A symmetric farsighted stable set consists of outcomes that are achieved with uniform fees. Such outcomes seem more plausible by the symmetry of firms. Therefore, we concentrate on analyzing symmetric farsighted stable sets.

4 Main results

4.1 The characterization

We state and prove a characterization of symmetric farsighted stable sets of the negotiation stage. We begin with some preparations. Let

$$A =]0, \max_{s=1,...,n}(W(s)-L(0))[.$$

For any $\alpha \in A$, define

$$B(\alpha) = \arg \max_{s=1,...,n} s(W(s) - L(0) - \alpha).$$

Note that A is the set of net profits of licensee firms where a patent holder and licensee firms are made better off than $x^{\emptyset}$. Therefore, A can be regarded as the set of strictly individually rational net profits of licensee firms since any of a patent holder and licensee firms can solely induce $(\emptyset, x^{\emptyset})$. Given $\alpha \in A$, $B(\alpha)$ is the set of the optimal numbers of licensee firms for a patent holder, provided that each licensee firm obtains the net profit α. Note that $A \neq \emptyset$ by Assumption 1 (i). Note also that $B(\alpha) \neq \emptyset$ for all $\alpha \in A$ by the finiteness of the firms.

For any $\alpha \in A$, define

$$\bar{X}(\alpha) = \{(S, x) \in \bar{X} | s \in B(\alpha), x_0 = s(W(s) - L(0) - \alpha), x_i = L(0) + \alpha \text{ for all } i \in S\}.$$

In $\bar{X}(\alpha)$, each licensee firm receives an identical net profit α. Such outcomes are yielded by a uniform fee for patent licensing since firms are symmetric. In $\bar{X}(\alpha)$, a patent holder determines the number of licensee firms as if it maximizes its own profit subject to the uniform profit of licensee firms. Note that when $s < n$ for some $s \in B(\alpha)$, $\bar{X}(\alpha)$ includes multiple outcomes where the members of licensee firms are different while the number of licensee firms is s. Note also that $B(\alpha)$ may include two or more natural numbers for a given $\alpha \in A$. Thus, $\bar{X}(\alpha)$ may include outcomes (S, x) and (S', x') such that $s \neq s'$. Anyway, we have $x_0 = x'_0$ and $x_i = x'_j$ for all $i \in S$ and all $j \in S'$ by the definitions of $B(\alpha)$ and $\bar{X}(\alpha)$.

Now, we turn to our main result that completely characterizes symmetric farsighted stable sets.

Theorem 1 *Let $\bar{K} \subseteq X$. Then, $\bar{K}$ is a symmetric farsighted stable set if and only if $\bar{K} = \bar{X}(\alpha)$ for some $\alpha \in A$.*

Proof. We first prove "if" part. Fix an arbitrary $\alpha^* \in A$. Let $S^* \subseteq N$ with $S^* \neq \emptyset$ such that $s^* \in B(\alpha^*)$. We first show internal stability. Fix arbitrary $(S, x), (S', x') \in \bar{X}(\alpha^*)$. Suppose that $(S', x') \succ (S, x)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S, x)$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $x'_i > z_i^{h-1}$ for all $i \in R^h$. By $x_0 = x'_0 = s^*(W(s^*) - L(0) - \alpha^*)$, $0 \notin R^1$. Then, $R^1 \subseteq S$ by Assumption 2. However, $x_i = L(0) + \alpha^* \geq x'_i$ for all $i \in S$ because $x'_i = L(0) + \alpha^*$ when $i \in S'$ and

$x'_i = L(s') < L(0)$ when $i \notin S'$. Thus, $R^1 = \emptyset$, contradicting that R^1 is a coalition. Hence, $\bar{X}(\alpha^*)$ is internally stable.

Next, we show external stability. Fix an arbitrary $(T, y) \in X \setminus \bar{X}(\alpha^*)$. Let (S^*, x^*) be a symmetric outcome such that $x^*_0 = s^*(W(s^*) - L(0) - \alpha^*)$, $x^*_i = L(0) + \alpha^*$ for all $i \in S^*$, and $x^*_i = L(s^*)$ for all $i \in N \backslash S^*$. Note that $(S^*, x^*) \in \bar{X}(\alpha^*)$. If $(T, y) = (\emptyset, x^\emptyset)$, then it is straightforward that $(T, y) = (\emptyset, x^\emptyset) \rightarrow_{\{0\}\cup S^*} (S^*, x^*)$ yields $(S^*, x^*) \succ (T, y)$. Thus, assume that $T \neq \emptyset$. We distinguish two cases.

Case 1. $y_0 < s^*(W(s^*) - L(0) - \alpha^*)$.

In this case,

$$(T, y) \rightarrow_{\{0\}} (\emptyset, x^\emptyset) \rightarrow_{\{0\}\cup S^*} (S^*, x^*),$$

yield $(S^*, x^*) \succ (T, y)$ by

- $x^*_0 = s^*(W(s^*) - L(0) - \alpha^*) > \max\{0, y_0\} = \max\{x^\emptyset_0, y_0\}$ and
- $x^*_i = L(0) + \alpha^* > L(0) = x^\emptyset_i$ for all $i \in S^*$.

Case 2. $y_0 \geq s^*(W(s^*) - L(0) - \alpha^*)$.

We claim that $y_i < L(0) + \alpha^*$ for some $i \in T$. Suppose that $y_i \geq L(0) + \alpha^*$ for all $i \in T$. Then,

$$tW(t) = \sum_{i\in\{0\}\cup T} y_i \geq s^*(W(s^*) - L(0) - \alpha^*) + t(L(0) + \alpha^*), \tag{1}$$

which is equivalent to

$$t(W(t) - L(0) - \alpha^*) \geq s^*(W(s^*) - L(0) - \alpha^*). \tag{2}$$

If $y_j > L(0) + \alpha^*$ for some $j \in T$, then (1) as well as (2) holds in a strict inequality, contradicting $s^* \in B(\alpha^*)$. Thus, $y_i = L(0) + \alpha^*$ for all $i \in T$. By $tW(t) = \sum_{i\in\{0\}\cup T} y_i$, $y_0 = t(W(t) - L(0) - \alpha^*)$. Then, $t \in B(\alpha^*)$ by (2) and $s^* \in B(\alpha^*)$. Therefore, $(T, y) \in \bar{X}(\alpha^*)$, contradicting $(T, y) \in X \setminus \bar{X}(\alpha^*)$. Hence, there exists some $j \in T$ such that $y_j < L(0) + \alpha^*$.

Let $(S', x') \in \bar{X}$ such that $j \in S'$, $s' \in B(\alpha^*)$, $x'_0 = s'(W(s') - L(0) - \alpha^*)$, $x'_i = L(0) + \alpha^*$ for all $i \in S'$, and $x'_i = L(s')$ for all $i \in N \setminus S'$. Note that $(S', x') \in \bar{X}(\alpha^*)$. Then,

$$(T, y) \rightarrow_{\{j\}} (\emptyset, x^\emptyset) \rightarrow_{\{0\}\cup S'} (S', x').$$

yield $(S', x') \succ (T, y)$ by

- $x'_j = L(0) + \alpha^* > y_j$,
- $x'_i = L(0) + \alpha^* > x^{\emptyset}_i$ for all $i \in S'$, and
- $x'_0 = s^*(W(s^*) - L(0) - \alpha^*) > 0 = x^{\emptyset}_0$.

Hence $\bar{X}(\alpha^*)$ is externally stable.

Next, we prove "only if" part. We begin with two lemmas that state properties of the indirect dominance relation.

Lemma 1 *For any $(S, x) \in \bar{X}$, $(S, x) \succ (\emptyset, x^{\emptyset})$ if and only if $x_0 > 0$ and $x_i > L(0)$ for all $i \in S$.*

Proof. The sufficiency is straightforward since $(\emptyset, x^{\emptyset}) \rightarrow_{\{0\}\cup S} (S, x)$ yields $(S, x) \succ (\emptyset, x^{\emptyset})$ by $x_0 > 0 = x^{\emptyset}_0$ and $x_i > L(0) = x^{\emptyset}_i$ for all $i \in S$.

We turn to the necessity. Fix an arbitrary $(S, x) \in \bar{X}$ such that $(S, x) \succ (\emptyset, x^{\emptyset})$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (\emptyset, x^{\emptyset})$, $(Q^m, z^m) = (S, x)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $x_i > z^{h-1}_i$ for all $i \in R^h$. Obviously, $(S, x) \neq (\emptyset, x^{\emptyset})$. Thus, $S \neq \emptyset$.

Suppose that either $x_0 \leq 0$ or $x_i \leq L(0)$ for some $i \in S$. The latter condition implies that $x_i \leq L(0)$ for all $i \in S$ by the symmetry of (S, x). If $x_0 \leq 0 = x^{\emptyset}_0$, then $R^1 \subseteq S$. If $x_i \leq L(0) = x^{\emptyset}_i$ for all $i \in S$, then $R^1 = \{0\}$ since $x^{\emptyset}_i = L(0) > L(s) = x_i$ for all $i \in N \setminus S$. In either case, $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. However, $(\emptyset, x^{\emptyset}) \rightarrow_T (\emptyset, x^{\emptyset})$ is impossible for any coalition T by Assumption 2. Hence, $x_0 > 0$ and $x_i > L(0)$ for all $i \in S$. ■

Note that this lemma is not retained with asymmetric outcomes. By employing the necessity of Lemma 1, we can prove the following lemma.

Lemma 2 *Let $(S, x) \in X \setminus \{(\emptyset, x^{\emptyset})\}$ and $(T, y) \in \bar{X} \setminus \{(\emptyset, x^{\emptyset})\}$ be outcomes such that $(T, y) \succ (S, x)$ and either $y_0 \leq 0$ or $y_i \leq L(0)$ for all $i \in T$. Let $(Q^0, z^0), ..., (Q^m, z^m)$ and $R^1, ..., R^m$ be sequences yielding $(T, y) \succ (S, x)$, i.e. $(Q^0, z^0) = (S, x)$, $(Q^m, z^m) = (T, y)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $y_i > z^{h-1}_i$ for all $i \in R^h$. Then, $(Q^h, z^h) \neq (\emptyset, x^{\emptyset})$ for all $h = 0, ..., m$.*

Proof. Let $(S, x) \in X \setminus \{(\emptyset, x^{\emptyset})\}$ and $(T, y) \in \bar{X} \setminus \{(\emptyset, x^{\emptyset})\}$ be outcomes such that $(T, y) \succ (S, x)$ and either $y_0 \leq 0$ or $y_i \leq L(0)$ for all $i \in T$. Suppose that there exists some $\ell = 0, ..., m$ such that $(Q^{\ell}, z^{\ell}) = (\emptyset, x^{\emptyset})$. Note that $0 < \ell < m$ by $(S, x) \neq (\emptyset, x^{\emptyset}) \neq (T, y)$. Then,

$$(Q^{\ell}, z^{\ell}) \rightarrow_{R^{\ell+1}} (Q^{\ell+1}, z^{\ell+1}) \rightarrow_{R^{\ell+2}} \cdots \rightarrow_{R^m} (Q^m, z^m)$$

yield $(T,y) = (Q^m, z^m) \succ (Q^\ell, z^\ell) = (\emptyset, x^\emptyset)$ since for all $h = \ell{+}1, ..., m$, $(Q^{h-1}, z^{h-1}) \to_{R^h} (Q^h, z^h)$ and $y_i > z_i^{h-1}$ for all $i \in R^h$. This contradicts the necessity of Lemma 1 by the choice of (T,y). Hence, $(Q^h, z^h) \neq (\emptyset, x^\emptyset)$ for all $h = 0, ..., m$. ■

Fix an arbitrary symmetric farsighted stable set $\bar{K}$. In what follows, we state and prove six claims. Claim 1 shows that the null outcome is never included in $\bar{K}$. By this claim and external stability of $\bar{K}$, there exists some $(\bar{S}, \bar{x}) \in \bar{K}$ with $\bar{S} \neq \emptyset$. We may choose $(\bar{S}, \bar{x})$ such that licensee firms receive the greatest payoffs among outcomes in $\bar{K}$ such that the patent holder's payoff is $\bar{x}_0$. Claims 2 and 3 give necessary conditions for an outcome in $\bar{K}$ if the patent holder's profit is different from $\bar{x}_0$. If (T,y) with $y_0 \neq \bar{x}_0$ is in $\bar{K}$, then these conditions yield that $y_0 > \bar{x}_0$, $\bar{S} \cap T = \emptyset$, and $y_i \leq L(\bar{s})$ for all $i \in N$. Moreover, by using those conditions, we show in Claim 4 that the patent holder's profit is actually identical at any outcome in $\bar{K}$. To be more precise, consider (S^*, x^*), where S^* is obtained by replacing a firm in $\bar{S}$ with a firm in T and any licensee firm enjoys the same payoff as a licensee firm at $(\bar{S}, \bar{x})$. Then, (S^*, x^*) indirectly dominates (T, y) because a firm in $S^* \cap T$ prefers (S^*, x^*) and induces the null outcome by resolving (T, y), and $\{0\} \cup S^*$ induces (S^*, x^*) afterward. Thus, $(S^*, x^*) \notin \bar{K}$. However, any outcome (S', x') in $\bar{K}$ does not indirectly dominate (S^*, x^*). When $x'_0 = \bar{x}_0$, the payoffs of the patent holder and licensee firms at (S^*, x^*) is at least as much payoffs as those at (S', x') by the choice of (S^*, x^*) and $(\bar{S}, \bar{x})$. Thus, no player would like to start the paths that yield $(S', x') \succ (S^*, x^*)$. When $x'_0 > \bar{x}_0$, only the patent holder prefers (S', x') to (S^*, x^*) by the consequence of Claims 2 and 3. Thus, only the patent holder is willing to start paths that yield $(S', x') \succ (S^*, x^*)$, but only the null outcome can come next. Lemma 2 showed that such paths cannot yield $(S', x') \succ (S^*, x^*)$ since any firm receives no more than $L(0)$ at (S', x'). Therefore, external stability fails if $\bar{K}$ includes an outcome where the patent holder's payoff is different from $\bar{x}_0$. Then, we show that the licensee firms' profits are identical across outcomes in $\bar{K}$ in Claim 5. In Claim 6, we show that the number of licensee firms at any outcome in $\bar{K}$ maximizes the patent holder's profit provided that the licensee firms' profits are identically $L(0) + \bar{\alpha}$.

Claim 1 $(\emptyset, x^\emptyset) \notin \bar{K}$.

Proof of Claim 1. Suppose that $(\emptyset, x^\emptyset) \in \bar{K}$. Let $(N, \hat{x}) \in X$ such that $\hat{x}_0 = n(W(n) - L(0) - \delta)$ and $\hat{x}_i = L(0) + \delta$ for all $i \in N$, where $\delta > 0$ is a sufficiently small real number so that $\hat{x}_0 > 0$. Note that we can take such $(N, \hat{x})$ by $W(n) > L(0)$. Then, $(N, \hat{x}) \succ (\emptyset, x^\emptyset)$ by Lemma 1. Thus, $(N, \hat{x}) \notin \bar{K}$ by internal stability of $\bar{K}$. It is also easy to see that $(\emptyset, x^\emptyset) \succ (N, \hat{x})$ is impossible by $\hat{x}_i > x_i^\emptyset$ for all $i \in \{0\} \cup N$. Thus, there exists some $(S, x) \in \bar{K}$ such that $S \neq \emptyset$ and $(S, x) \succ (N, \hat{x})$ by external stability of $\bar{K}$.

If $x_i < x_i^\emptyset$ for some $i \in \{0\} \cup S$, then $(\emptyset, x^\emptyset) \succ (S, x)$ holds by $(S, x) \rightarrow_{\{i\}} (\emptyset, x^\emptyset)$, contradicting internal stability of $\bar{K}$. Thus, $x_0 \geq 0$ and $x_i \geq L(0)$ for all $i \in S$. Moreover, by internal stability of $\bar{K}$ and the sufficiency of Lemma 1, either $x_0 = 0$ or $x_i = L(0)$ for all $i \in S$. Note that $x_i = L(s) < L(0)$ for all $i \in N \setminus S$.

By $(S, x) \succ (N, \hat{x})$, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (N, \hat{x})$, $(Q^m, z^m) = (S, x)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $x_i > z_i^{h-1}$ for all $i \in R^h$. If $x_0 = 0$, then $0 \notin R^1$ by $x_i > z_i^0 = \hat{x}_i$ for all $i \in R^1$. If $x_i = L(0)$ for all $i \in S$, then $R^1 = \{0\}$ since $\hat{x}_i > L(0) = x_i$ for all $i \in S$ and $\hat{x}_i > L(0) > L(s) = x_i$ for all $i \in N \setminus S$, while it is assumed that $x_i > z_i^0 = \hat{x}_i$ for all $i \in R^1$. Therefore, $(Q^1, z^1) = (\emptyset, x^\emptyset)$ in either case. This contradicts Lemma 2 since either $x_0 = 0$ or $x_i = L(0)$ for all $i \in S$. Hence $(\emptyset, x^\emptyset) \notin \bar{K}$. □

By Lemma 1, Claim 1, and external stability of $\bar{K}$, there exists some $(\bar{S}, \bar{x}) \in \bar{K}$ such that $\bar{S} \neq \emptyset$, $\bar{x}_0 > 0$ and $\bar{x}_i > L(0)$ for all $i \in \bar{S}$. Without loss of generality, we may choose $(\bar{S}, \bar{x})$ so that for any $(S', x') \in \bar{K}$ $x'_0 = \bar{x}_0$ implies $x'_j \leq \bar{x}_i$ for all $j \in S'$ and $i \in \bar{S}$. We can take such an outcome since $\{(T, y) \in \bar{X} | y_0 = \bar{x}_0\}$ is finite. In words, $(\bar{S}, \bar{x})$ gives the largest payoffs for the licensee firms among the outcomes in $\bar{K}$ that guarantee the same patent holder's payoffs. Denote $\bar{x}_i = L(0) + \bar{\alpha}$ for all $i \in \bar{S}$. Note that $\bar{\alpha} \in A$ by $\bar{s}(W(\bar{s}) - L(0) - \bar{\alpha}) = \bar{x}_0 > 0$. Throughout this proof, this $(\bar{S}, \bar{x})$ is fixed.

Claim 2 *For any $(T, y) \in \bar{K}$, $y_0 \neq \bar{x}_0$ implies (i) $y_0 > \bar{x}_0$, (ii) $\bar{S} \cap T = \emptyset$, and (iii) $y_i = L(\bar{s})$ for all $i \in T$.*

Proof of Claim 2. Fix an arbitrary $(T, y) \in \bar{K}$. Assume that $y_0 \neq \bar{x}_0$. Note that both $(\bar{S}, \bar{x})$ and (T, y) are symmetric. First, suppose that $y_0 < \bar{x}_0$. Then, $(T, y) \rightarrow_{\{0\}} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (T, y)$, contradicting internal stability of $\bar{K}$. Hence, we obtain (i) $y_0 > \bar{x}_0$.

Second, suppose that $\bar{S} \cap T \neq \emptyset$. If $\bar{x}_i > y_i$ for all $i \in \bar{S} \cap T$, then $(T, y) \rightarrow_{\bar{S} \cap T} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (T, y)$ by $\bar{x}_0 > 0 = x_0^\emptyset$ and $\bar{x}_i > L(0) = x_i^\emptyset$ for all $i \in \bar{S}$. This contradicts internal stability of $\bar{K}$. Assume, therefore, that $\bar{x}_i \leq y_i$ for all $i \in \bar{S} \cap T$. Then, $y_i \geq \bar{x}_i > L(0)$ for all $i \in \bar{S} \cap T$. By the symmetry of (T, y), $y_i > L(0)$ for all $i \in T$. Thus, $(\bar{S}, \bar{x}) \rightarrow_{\{0\}} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup T} (T, y)$ yield $(T, y) \succ (\bar{S}, \bar{x})$ by $y_0 > \bar{x}_0 > 0 = x_0^\emptyset$, contradicting internal stability of $\bar{K}$. Hence, we have (ii) $\bar{S} \cap T = \emptyset$.

Finally, we show (iii). If $y_i > L(\bar{s})$ for all $i \in T$, then $(\bar{S}, \bar{x}) \rightarrow_{\{0\}\cup T} (T, y)$ yields $(T, y) \succ (\bar{S}, \bar{x})$ by

- $y_0 > \bar{x}_0$ and
- $\bar{x}_i = L(\bar{s}) < y_i$ for all $i \in T$,

where the latter follows from (ii) $\bar{S} \cap T = \emptyset$. If $y_i < L(\bar{s})$ for all $i \in T$, then $(T, y) \rightarrow_T (\emptyset, x^{\emptyset}) \rightarrow_{\{0\}\cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (T, y)$ by

- $\bar{x}_0 > 0 = x_0^{\emptyset}$,
- $\bar{x}_i > L(0) = x_i^{\emptyset}$ for all $i \in \bar{S}$, and
- $\bar{x}_i = L(\bar{s}) > y_i$ for all $i \in T$,

where the third statement follows from (ii) $\bar{S} \cap T = \emptyset$. Either case contradicts internal stability of $\bar{K}$. Hence, we have (iii) $y_i = L(\bar{s})$ for all $i \in T$. □

Claim 3 *Let $(T, y) \in \bar{K}$. If $y_0 \neq \bar{x}_0$, then $L(t) \leq L(\bar{s})$. Thus, $y_i \leq L(\bar{s})$ for all $i \in N$.*

Proof of Claim 3. Let $(T, y) \in \bar{K}$ such that $y_0 \neq \bar{x}_0$. Note that $y_0 > \bar{x}_0$, $\bar{S} \cap T = \emptyset$, and $y_i = L(\bar{s})$ for all $i \in T$ by Claim 2. Suppose that $L(t) > L(\bar{s})$. Without loss of generality, we may assume that $L(t) \geq L(t')$ for any symmetric outcome $(T', y') \in \bar{K}$ such that $y'_0 \neq \bar{x}_0$.

Let $(T, \hat{y})$ be a symmetric outcome such that $\hat{y}_i = y_i + \hat{\varepsilon} = L(\bar{s}) + \hat{\varepsilon}$ for all $i \in T$ and $\hat{y}_i = L(t) = y_i$ for all $i \in N \setminus T$, where $\hat{\varepsilon} > 0$ is sufficiently small so that $\hat{y}_0 = y_0 - t\hat{\varepsilon} > \bar{x}_0$ and $\hat{y}_i = L(\bar{s}) + \hat{\varepsilon} < L(t) < L(0)$ for all $i \in T$. Then, $(T, \hat{y}) \notin \bar{K}$ by Claim 2(iii).

We show that neither $(\bar{S}, \bar{x}) \succ (T, \hat{y})$ nor $(T, y) \succ (T, \hat{y})$. First, suppose that $(\bar{S}, \bar{x}) \succ (T, \hat{y})$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (T, \hat{y})$, $(Q^m, z^m) = (\bar{S}, \bar{x})$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < \bar{x}_i$ for all $i \in R^h$. By $\bar{x}_0 < \hat{y}_0$, $0 \notin R^1$. Then, $R^1 \subseteq T$ by Assumption 2. On the other hand, $\bar{x}_i = L(\bar{s}) < \hat{y}_i$ for all $i \in T$ by $\bar{S} \cap T = \emptyset$. Thus, $R^1 = \emptyset$, contradicting that R^1 is a coalition. Hence, $(\bar{S}, \bar{x}) \succ (T, \hat{y})$ is impossible.

Next, suppose that $(T, y) \succ (T, \hat{y})$. Then, there exist sequences of outcomes (Q^0, z^0), ..., (Q^m, z^m) and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (T, \hat{y})$, $(Q^m, z^m) = (T, y)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < y_i$ for all $i \in R^h$. By $\hat{y}_i \geq y_i$ for all $i \in N$ and the nonemptiness of R^1, $R^1 = \{0\}$. Then, $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. This contradicts Lemma 2 by $y_i = L(\bar{s}) < L(0)$ for all $i \in T$. Hence, $(T, y) \succ (T, \hat{y})$ is also impossible.

By external stability of $\bar{K}$, there exists some $(S', x') \in \bar{K}$ such that $(S', x') \succ (T, \hat{y})$. Note that $S' \neq \emptyset$ by Claim 1. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (T, \hat{y})$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < x'_i$ for all $i \in R^h$.

Suppose that $x'_0 = \bar{x}_0$. By $x'_0 = \bar{x}_0 < \hat{y}_0$, $0 \notin R^1$. Then, $R^1 \subseteq T$ and $(Q^1, z^1) = (\emptyset, x^{\emptyset})$ by Assumption 2. Then, $(\emptyset, x^{\emptyset}) \rightarrow_{R^2} (Q^2, z^2) \rightarrow_{R^3} \cdots \rightarrow_{R^m} (S', x')$ yield $(S', x') \succ (\emptyset, x^{\emptyset})$ by the choices of the outcomes and coalitions constituting these paths. It must be $x'_i > L(0)$ for all $i \in S'$ by Lemma 1. If both $R^1 \setminus S' \neq \emptyset$ and $L(s') \leq L(\bar{s}) + \hat{\varepsilon}$, then $x'_j = L(s') \leq L(\bar{s}) + \hat{\varepsilon} = \hat{y}_j$ for all $j \in R^1 \setminus S' \subseteq T \setminus S'$. This contradicts that $x'_i > \hat{y}_i$ for all $i \in R^1$. Thus, $R^1 \subseteq S'$ or $L(s') > L(\bar{s}) + \hat{\varepsilon}$. Assume that $R^1 \subseteq S'$. Note that $R^1 \subseteq S' \cap T$. Then, $(T, y) \rightarrow_{R^1} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup S'} (S', x')$ yield $(S', x') \succ (T, y)$ by

- $x'_0 = \bar{x}_0 > 0 = x_0^{\emptyset}$,
- $x'_i > L(0) = x_i^{\emptyset}$ for all $i \in S'$, and
- $x'_i > L(0) > L(\bar{s}) = y_i$ for all $i \in R^1$, where $R^1 \subseteq S' \cap T$.

This contradicts internal stability of $\bar{K}$. Assume, therefore, that $L(s') > L(\bar{s}) + \hat{\varepsilon}$. Then,

$$x'_i = L(s') > L(\bar{s}) + \hat{\varepsilon} > L(\bar{s}) = y_i \text{ for all } i \in T \setminus S'.$$

Together with $x'_i > L(0) > L(\bar{s}) = y_i$ for all $i \in S' \cap T$, we obtain $x'_i > y_i$ for all $i \in T$. Then, $(T, y) \rightarrow_T (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup S'} (S', x')$ yield $(S', x') \succ (T, y)$ by $x'_0 = \bar{x}_0 > 0 = x_0^{\emptyset}$ and $x'_i > L(0) = x_i^{\emptyset}$ for all $i \in S'$. This contradicts internal stability of $\bar{K}$. Hence, $x'_0 \neq \bar{x}_0$.

By Claim 2, $x'_0 > \bar{x}_0$, $\bar{S} \cap S' = \emptyset$, and $x'_i = L(\bar{s})$ for all $i \in S'$. Note that $L(t) \geq L(s')$ since (T, y) was chosen so that $L(t) \geq L(t')$ for any $(T', y') \in \bar{K}$ with $y'_0 \neq \bar{x}_0$. If there exists some $\ell = 1, ..., m$ such that $0 \notin R^{\ell}$ or $R^{\ell} = \{0\}$, then $(Q^{\ell}, z^{\ell}) = (\emptyset, x^{\emptyset})$. This contradicts Lemma 2 by $x'_i < L(0)$ for all $i \in S'$. Hence,

$$0 \in R^h \text{ and } R^h \setminus \{0\} \neq \emptyset \text{ for all } h = 1, ..., m. \tag{3}$$

For any $i \notin \{0\} \cup T$, we have that

$$\hat{y}_i = L(t) > L(\bar{s}) = x'_i \text{ if } i \in S';$$
$$\hat{y}_i = L(t) \geq L(s') = x'_i \text{ if } i \notin S'.$$

Therefore, $R^1 \subseteq \{0\} \cup T$, and thus, $Q^1 = R^1 \setminus \{0\} \subseteq T$. Note that $(\bar{S}, \bar{x}) \to_{R^1} (Q^1, z^1)$ by $0 \in R^1$. Since $(T, \hat{y}) \to_{R^1} (Q^1, z^1) \to_{R^2} \cdots \to_{R^m} (S', x')$ yield $(S', x') \succ (T, \hat{y})$, $x'_i > \hat{y}_i$ for all $i \in R^1 = \{0\} \cup Q^1$. By $\bar{S} \cap Q^1 \subseteq \bar{S} \cap T = \emptyset$, $\bar{x}_i = L(\bar{s}) < \hat{y}_i < x'_i$ for all $i \in Q^1$. By the choice of $\hat{\varepsilon}$, $\bar{x}_0 < \hat{y}_0 < x'_0$. Therefore,

$$(\bar{S}, \bar{x}) \to_{R^1} (Q^1, z^1) \to_{R^2} \cdots \to_{R^m} (S', x')$$

yield $(S', x') \succ (\bar{S}, \bar{x})$. This contradicts internal stability of $\bar{K}$. Hence, $L(t) \leq L(\bar{s})$. The last statement immediately follows from this and Claim 2(iii). □

Claim 4 *For any* $(T, y) \in \bar{K}$, $y_0 = \bar{x}_0$

Proof of Claim 4. Suppose that there exists some $(T, y) \in \bar{K}$ such that $y_0 \neq \bar{x}_0$. Note that $T \neq \emptyset$ by Claim 1. Note also that (T, y) is symmetric by the definition of $\bar{K}$. By Claim 2, $y_0 > \bar{x}_0$, $\bar{S} \cap T = \emptyset$, and $y_i = L(\bar{s})$ for all $i \in T$. By Claim 3, $L(t) \leq L(\bar{s})$.

Let $S^* \subseteq N$ be a coalition such that $s^* = \bar{s}$, $S^* \subseteq \bar{S} \cup T$, and $S^* \neq \bar{S}$. Note that we can take such S^* by $\bar{S} \cap T = \emptyset$ and $\bar{S}, T \neq \emptyset$. Note also that $S^* \cap T \neq \emptyset$ by the choice of S^*. Let (S^*, x^*) be a symmetric outcome such that

$$x^*_i = \begin{cases} \bar{x}_0 = \bar{s}(W(\bar{s}) - L(0) - \bar{\alpha}) & \text{if } i = 0; \\ L(0) + \bar{\alpha} & \text{if } i \in S^*; \\ L(\bar{s}) & \text{if } i \in N \setminus S^*. \end{cases}$$

Note that (S^*, x^*) satisfies the definition of an outcome by $s^* = \bar{s}$. Since either $y_i = L(\bar{s})$ or $y_i = L(t)$ for all $i \in N$, $x^*_i > L(0) > y_i$ for all $i \in S^*$ by Assumption 1(b). Then, $(T, y) \to_{S^* \cap T} (\emptyset, x^{\emptyset}) \to_{\{0\} \cup S^*} (S^*, x^*)$ yield $(S^*, x^*) \succ (T, y)$ by

- $x^*_i > y_i$ for all $i \in S^* \cap T$, where $S^* \cap T \neq \emptyset$,
- $x^*_0 = \bar{x}_0 > 0 = x^{\emptyset}_0$, and
- $x^*_i > L(0) = x^{\emptyset}_i$ for all $i \in S^*$.

Thus, $(S^*, x^*) \notin \bar{K}$ by internal stability of $\bar{K}$.

By the symmetry and external stability of $\bar{K}$, there exists a symmetric $(S', x') \in \bar{K}$ such that $(S', x') \succ (S^*, x^*)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S^*, x^*)$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \to_{R^h} (Q^h, z^h)$ and $z^{h-1}_i < x'_i$ for all $i \in R^h$. By Claim 2, $x'_0 \geq \bar{x}_0$.

Suppose that $x'_0 = \bar{x}_0 = x^*_0$. Recall that $(\bar{S}, \bar{x})$ is chosen so that for any $(S'', x'') \in \bar{K}$, $\bar{x}_0 = x''_0$ implies $x''_j \leq \bar{x}_i$ for all $j \in S''$ and $i \in \bar{S}$. Then, by the definition of (S^*, x^*), $x'_0 = \bar{x}_0 = x^*_0$ implies $x'_j \leq x^*_i$ for all $j \in S'$ and $i \in S^*$. By $x'_0 = \bar{x}_0 = x^*_0$, $0 \notin R^1$, and thus, $R^1 \subseteq S^*$ by Assumption 2. Therefore, $x'_i > x^*_i = L(0) + \bar{\alpha}$ for all $i \in R^1 \subseteq S^*$. However, for all $i \in R^1 \subseteq S^*$, $x'_i = L(s') < L(0) < x^*_i$ if $i \notin S'$, while $x'_i \leq x^*_i$ if $i \in S'$. This is a contradiction.

Assume, therefore, that $x'_0 > \bar{x}_0$. By the choice of (S^*, x^*), $x^*_i \geq L(\bar{s})$ for all $i \in N$. On the other hand, $x'_i \leq L(\bar{s})$ for all $i \in N$ by the last statement of Claim 3. Therefore, $R^1 = \{0\}$ and $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. This contradicts Lemma 2 since we have $x'_i = L(\bar{s}) < L(0)$ for all $i \in S'$ by Claim 2(iii). Hence, $\bar{x}_0 = y_0$. □

Claim 5 *For any* $(S', x') \in \bar{K}$, $x'_i = L(0) + \bar{\alpha}$ *for all* $i \in S'$.

Proof of Claim 5. Fix an arbitrary $(S', x') \in \bar{K}$, which is symmetric. Suppose that $x'_j \neq L(0) + \bar{\alpha}$ for all $j \in S'$. Note that $S' \neq \emptyset$ by Claim 1. Recall again that $(\bar{S}, \bar{x})$ is chosen so that for any $(S'', x'') \in \bar{K}$, $\bar{x}_0 = x''_0$ implies $x''_j \leq \bar{x}_i$ for all $j \in S''$ and $i \in \bar{S}$. Then, by Claim 4, $x'_j < \bar{x}_i = L(0) + \bar{\alpha}$ for all $i \in \bar{S}$ and $j \in S'$.

Suppose that $\bar{S} \cap S' \neq \emptyset$. Then, $(S', x') \rightarrow_{\bar{S} \cap S'} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (S', x')$ by

- $x'_i < \bar{x}_i$ for all $i \in \bar{S} \cap S' \neq \emptyset$,
- $\bar{x}_0 > 0 = x^{\emptyset}_0$, and
- $\bar{x}_i > L(0) = x^{\emptyset}_i$ for all $i \in \bar{S}$.

This contradicts internal stability of $\bar{K}$. Therefore, assume that $\bar{S} \cap S' = \emptyset$.

Let (S^*, x^*) be a symmetric outcome such that $s^* = \bar{s}$, $S' \cap S^* \neq \emptyset$, $S^* \neq \bar{S}$, and

$$x^*_i = \begin{cases} \bar{s}(W(\bar{s}) - L(0) - \bar{\alpha}) & \text{if } i = 0; \\ L(0) + \bar{\alpha} & \text{if } i \in S^*; \\ L(\bar{s}) & \text{if } i \in N \setminus S^*. \end{cases}$$

Note that (S^*, x^*) satisfies the definition of an outcome by $s^* = \bar{s}$. Note also that we can take such S^* by replacing a firm in $\bar{S}$ with a firm in S' since $\bar{S} \cap S' = \emptyset$, where $\bar{S}, S' \neq \emptyset$. Moreover, $x^*_0 = \bar{x}_0$ and $x^*_i = \bar{x}_j$ for all $i \in S^*$ and $j \in \bar{S}$. Then, $(S', x') \rightarrow_{S' \cap S^*} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup S^*} (S^*, x^*)$ yield $(S^*, x^*) \succ (S', x')$ by

- $x_i^* = L(0) + \bar{\alpha} > x_i'$ for all $i \in S' \cap S^*$,
- $x_0^* = \bar{x}_0 > 0 = x_0^{\emptyset}$, and
- $x_i^* = L(0) + \bar{\alpha} > L(0) = x_i^{\emptyset}$ for all $i \in S^*$.

Thus, $(S^*, x^*) \notin \bar{K}$ by internal stability of $\bar{K}$.

By external stability of $\bar{K}$, there exists some $(\hat{S}, \hat{x}) \in \bar{K}$ such that $(\hat{S}, \hat{x}) \succ (S^*, x^*)$. Thus, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S^*, x^*)$, $(Q^m, z^m) = (\hat{S}, \hat{x})$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < \hat{x}_i$ for all $i \in R^h$. By Claim 4, $\hat{x}_0 = \bar{x}_0 = x_0^*$. Thus, $0 \notin R^1$ and $R^1 \subseteq S^*$. By the choice of $(\bar{S}, \bar{x})$ and the definition of (S^*, x^*),

$$\hat{x}_i \leq L(x) + \bar{\alpha} = x_i^* \text{ if } i \in S^* \cap \hat{S};$$
$$\hat{x}_i < L(0) < L(0) + \bar{\alpha} = x_i^* \text{ if } i \in S^* \setminus \hat{S}.$$

Thus, $S^* \cap R^1 = \emptyset$ that implies $R^1 = \emptyset$, contradicting that R^1 is a coalition. Hence, $x_j' = \bar{x}_i$ for all $i \in \bar{S}$ and $j \in S'$. □

Claim 6 $\bar{s} \in B(\bar{\alpha})$.

Proof of Claim 6. Suppose that $\bar{s} \notin B(\bar{\alpha})$. Let $(S^*, x^*) \in \bar{X}$ such that $s^* \in B(\bar{\alpha})$ and

$$x_i^* = \begin{cases} s^*(W(s^*) - L(0) - \bar{\alpha}) & \text{if } i = 0; \\ L(0) + \bar{\alpha} & \text{if } i \in S^*; \\ L(s^*) & \text{if } i \in N \setminus S^*. \end{cases}$$

Note that $x_0^* > \bar{x}_0$ by the definition of $B(\bar{\alpha})$. Then, $(S^*, x^*) \notin \bar{K}$ by Claim 4. By external stability of $\bar{K}$, there exists some $(S', x') \in \bar{K}$ such that $(S', x') \succ (S^*, x^*)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S^*, x^*)$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < x_i'$ for all $i \in R^h$. Note that $x_0' = \bar{x}_0 < x_0^*$ by Claim 4. Thus, $0 \notin R^1$. Then, $R^1 \subseteq S^*$ by Assumption 2. Note that $x_i' = L(0) + \bar{\alpha}$ for all $i \in S'$ by Claim 5. For any $i \in S^*$,

$$x_i^* = L(0) + \bar{\alpha} = x_i' \text{ if } i \in S';$$
$$x_i^* = L(0) + \bar{\alpha} > L(s') = x_i' \text{ if } i \notin S'.$$

Thus, $R^1 \cap S^* = \emptyset$ that implies $R^1 = \emptyset$ by $R^1 \subseteq S^*$. This contradicts that R^1 is a coalition. Hence, $\bar{s} \in B(\bar{\alpha})$. □

By Claim 4-6, $\bar{K} \subseteq \bar{X}(\bar{\alpha})$. We have already proved that $\bar{X}(\bar{\alpha})$ is a symmetric farsighted stable set. Then, we obtain $\bar{K} = \bar{X}(\bar{\alpha})$ by internal stability of $\bar{X}(\bar{\alpha})$ and external stability of $\bar{K}$. ■

Note that there are only two steps of an objection and a counter-objection in the proofs of "if" part of Theorem 1, even if we assume that all players are farsighted.

The following corollary is immediate from Theorem 1, $A \neq \emptyset$, and $B(\alpha) \neq \emptyset$ for any $\alpha \in A$.

Corollary 1 *A symmetric farsighted stable set exists.*

4.2 The relationship with the core

Farsighted stable sets yielding single payoffs have been well investigated and shown to have close relationship with the cores. In our model, any outcome in the relative interior of the core solely constitutes a farsighted stable set even if the outcome is not symmetric. However, the converse does not necessarily hold. We begin with the definitions of the core and its relative interior.

Definition 3

- *We say that an outcome $(S, x) \in X$ is in the core if there exist no coalition $Q \subseteq \{0\} \cup N$ and $(T, y) \in X$ such that $(S, x) \rightarrow_Q (T, y)$ and $y_i > x_i$ for all $i \in Q$.*
- *We say that an outcome $(S, x) \in X$ is in the relative interior of the core if there exist no coalition $Q \subseteq \{0\} \cup N$ and $(T, y) \in X$ such that $(T, y) \neq (S, x)$, $(S, x) \rightarrow_Q (T, y)$, and $y_i \geq x_i$ for all $i \in Q$.*

We denote C the core and $\mathring{C}$ the relative interior of the core.

The following proposition characterizes a condition for the nonemptiness of $\mathring{C}$. Similar results are shown by Watanabe and Muto (2008) and Hirai and Watanabe (2018), though the models are slightly different from the present paper, and they showed a condition for the nonemptiness of cores.

Proposition 1 *The relative interior of the core is nonempty if and only if $\{n\} = B(\alpha)$ for some $\alpha \in A$. Moreover, $(S, x) \in \mathring{C}$ implies that $S = N$, $x_0 > 0$, and $x_i > L(0)$ for all $i \in S$.*

Proof. We begin with the latter part. It is easy to see that $(\emptyset, x^{\emptyset}) \notin \mathring{C}$ because $(\emptyset, x^{\emptyset}) \rightarrow_{\{0\}\cup N} (N, x)$, where $x_0 = n(W(n) - L(0)) > 0 = x_0^{\emptyset}$ and $x_i = L(0)$ for all $i \in N$ by Assumption 1(i). Fix an arbitrary $(\tilde{S}, \tilde{x}) \in \mathring{C}$, where $\tilde{S} \neq \emptyset$. If $\tilde{x}_0 \leq 0$, then $(\tilde{S}, \tilde{x}) \rightarrow_{\{0\}} (\emptyset, x^{\emptyset})$ and $\tilde{x}_0 \leq 0 = x_0^{\emptyset}$, contradicting that $(\tilde{S}, \tilde{x}) \in \mathring{C}$. Thus, $\tilde{x}_0 > 0$. If $\tilde{x}_i \leq L(0)$ for some $i \in \tilde{S}$, then $(\tilde{S}, \tilde{x}) \rightarrow_{\{i\}} (\emptyset, x^{\emptyset})$ and $\tilde{x}_i \leq L(0) = x_i^{\emptyset}$, contradicting that $(\tilde{S}, \tilde{x}) \in \mathring{C}$. Thus, $\tilde{x}_i > L(0)$ for all $i \in \tilde{S}$. Next, suppose that $\tilde{S} \neq N$. By $\tilde{S} \neq \emptyset$, we can pick $j \in \tilde{S}$ and $j' \in N \setminus \tilde{S}$. Note that $\tilde{x}_{j'} = L(\tilde{s})$. Define $\tilde{T} = (\tilde{S} \setminus \{j\}) \cup \{j'\}$. Define $\tilde{y} \in \mathbb{R}^{n+1}$ such that $\tilde{y}_i = \tilde{x}_i$ for all $i \in \{0\} \cup (\tilde{S} \setminus \{j\})$, $\tilde{y}_i = L(\tilde{t})$ for all $i \in N \setminus \tilde{T}$, and $\tilde{y}_{j'} = \tilde{x}_j > L(0)$. Then, $(\tilde{S}, \tilde{x}) \rightarrow_{\{0\}\cup\tilde{T}} (\tilde{T}, \tilde{y})$, $\tilde{y}_i = \tilde{x}_i$ for all $i \in \{0\} \cup (\tilde{T} \setminus \{j'\})$, and $\tilde{y}_{j'} > L(0) > L(\tilde{s}) = \tilde{x}_{j'}$. This contradicts that $(\tilde{S}, \tilde{x}) \in \mathring{C}$. Hence, $\tilde{S} = N$.

We turn to the former part. First, assume that there exists some $\alpha^* \in A$ such that $\{n\} = B(\alpha^*)$. Let $x^* \in \mathbb{R}^{n+1}$ be such that $x_0^* = n(W(n) - L(0) - \alpha^*)$ and $x_i^* = L(0) + \alpha^*$ for all $i \in N$. Note that $(N, x^*) \in X$. We show that $(N, x^*) \in \mathring{C}$. Suppose that $(N, x^*) \notin \mathring{C}$. Then, there exist a nonempty $R \subseteq \{0\} \cup N$ and $(Q, z) \in X$ such that $(N, x^*) \rightarrow_R (Q, z)$ and $z_i \geq x_i^*$ for all $i \in R$. Then, $0 \in R$ since $0 \notin R$ implies that $(Q, z) = (\emptyset, x^{\emptyset})$, and thus, $z_i < x_i^*$ for all $i \in R$. Note that $Q = R \setminus \{0\}$. Then,

$$z_0 = qW(q) - \sum_{i \in Q} z_i \leq q(W(q) - L(0) - \alpha^*) < n(W(n) - L(0) - \alpha^*) = x_0^*$$

by $\{n\} = B(\alpha^*)$. This contradicts $z_0 \geq x_0^*$. Hence, $(N, x^*) \in \mathring{C}$.

Next, assume that $\{n\} \neq B(\alpha)$ for any $\alpha \in A$. We show that $\mathring{C} = \emptyset$. Suppose that there exists some $(S, y) \in \mathring{C}$. Note that we have already shown that $S = N$, $y_0 > 0$, and $y_i > L(0)$ for all $i \in N$ in the first part of this proof. Define $\alpha' = \left(\sum_{i \in N} y_i - nL(0)\right)/n$. Then, $\alpha' \in A$ since

$$n(W(n) - L(0) - \alpha') = nW(n) - nL(0) - \left(\sum_{i \in N} y_i - nL(0)\right) = nW(n) - \sum_{i \in N} y_i = y_0 > 0.$$

By $\{n\} \neq B(\alpha')$, there exists some $t = 1, ..., n-1$ such that $t \in B(\alpha')$. Let $\rho : N \rightarrow N$ be a permutation of N such that $y_{\rho(1)} \leq y_{\rho(2)} \leq \cdots \leq y_{\rho(n)}$. Let $T = \{\rho(1), ..., \rho(t)\}$. Define $z \in \mathbb{R}^{n+1}$ such that $z_{\rho(i)} = y_{\rho(i)}$ for all $i = 1, ..., t$, $z_{\rho(j)} = L(t)$ for all $j = t+1, ..., n$, and $z_0 = tW(t) - \sum_{i=1}^{t} z_{\rho(i)}$. Then, $(T, z) \in X$ and $(N, y) \rightarrow_{\{0\}\cup T} (T, z)$. We claim that $z_0 \geq y_0$. Note that

$$\sum_{i=1}^{t} z_{\rho(i)} = \sum_{i=1}^{t} y_{\rho(i)} \leq \frac{t \sum_{i \in N} y_i}{n} = t(L(0) + \alpha').$$

Then,

$$z_0 = tW(t) - \sum_{i=1}^{t} z_{\rho(i)} \geq t(W(t) - L(0) - \alpha') \geq n(W(n) - L(0) - \alpha') = y_0$$

by $t \in B(\alpha')$. This contradicts that $(N, y) \in \mathring{C}$ together with $z_{\rho(i)} = y_{\rho(i)}$ for all $i = 1, ..., t$. Hence, $\mathring{C} = \emptyset$. ■

Recall that $\bar{X}(\alpha)$ is a singleton if and only if $\{n\} = B(\alpha)$ because $\bar{X}(\alpha)$ includes multiple outcomes with an identical number of licensee firms when $s \in B(\alpha)$ for some $s < n$. It follows from Theorem 1 that a symmetric and singleton farsighted stable set exists if and only if $\{n\} = B(\alpha)$. Therefore, the necessary and sufficient condition for the nonemptiness of the relative interior of the core is same as that for the existence of a symmetric and singleton farsighted stable set.

Next, we show that the relative interior of the core is a subset of the union of singleton farsighted stable sets.

Proposition 2 *For any $(N, x) \in \mathring{C}$, $\{(N, x)\}$ is a farsighted stable set.*

Proof. Fix an arbitrary $(N, x) \in \mathring{C}$. Note that $x_0 > 0$ and $x_i > L(0)$ for all $i \in N$ by the latter part of Proposition 1. To prove that $\{(N, x)\}$ is a farsighted stable set, it suffices to show its external stability. It is straightforward that $(N, x) \succ (\emptyset, x^{\emptyset})$ by $(\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup N} (N, x)$, $x_0 > 0 = x_0^{\emptyset}$, and $x_i > L(0) = x_i^{\emptyset}$ for all $i \in N$. Fix an arbitrary $(T, y) \in X \setminus \{(N, x), (\emptyset, x^{\emptyset})\}$. Note that $T \neq \emptyset$ by $(T, y) \neq (\emptyset, x^{\emptyset})$. By Assumption 2(ii), $(N, x) \rightarrow_{\{0\} \cup T} (T, y)$. By $(N, x) \in \mathring{C}$, there exists some $j \in \{0\} \cup T$ such that $x_j > y_j$. Then, $(T, y) \rightarrow_{\{j\}} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup N} (N, x)$ yield $(N, x) \succ (T, y)$ by $x_j > y_j$, $x_0 > 0 = x_0^{\emptyset}$, and $x_i > L(0) = x_i^{\emptyset}$ for all $i \in N$. Thus, $\{(N, x)\}$ is externally stable. Hence, $\{(N, x)\}$ is a farsighted stable set. ■

Note that Proposition 2 also shows that the existence of an asymmetric farsighted stable set because the relative interior of the core usually includes an asymmetric outcome, though Theorem 1 says nothing for asymmetric farsighted stable set.

We turn to showing that there may exist a farsighted stable set $\{(N, x)\}$ such that (N, x) is not even in the core via an example, which is not symmetric.

Example 1 Let $N = \{1, 2\}$. Assume that $2(W(2) - L(0)) > W(1) - L(0) > 0$. Let $\alpha > 0$ be a sufficiently small real number such that $2(W(2) - L(0)) > W(1) - L(0) + \alpha$ and $L(0) - \alpha > L(1)$. Define $x^* = (2(W(2) - L(0)), L(0) + \alpha, L(0) - \alpha)$. Obviously,

$(N, x^*) \in X \setminus \bar{X}$. We have that $(N, x^*) \notin C$ because $(N, x^*) \to_{\{2\}} (\emptyset, x^\emptyset)$ and $x_2^\emptyset = L(0) > x_2^*$.

We show that $\{(N, x^*)\}$ is a farsighted stable set. Internal stability is obvious since it is a singleton. Thus, we show its external stability. We first show that $(N, x^*) \succ (\emptyset, x^\emptyset)$. This indirect dominance relation is yielded by $(\emptyset, x^\emptyset) \to_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \to_{\{0,1,2\}} (N, x^*)$ because $x_0^* = 2(W(2) - L(0)) > W(1) - L(0) > 0 = x_0^\emptyset$, $x_1^* = L(0) + \alpha > L(0) = x_1^\emptyset$, and $x_2^* = L(0) - \alpha > L(1)$. Hence, $(N, x^*) \succ (\emptyset, x^\emptyset)$.

Fix an arbitrary $(S, x) \in X \setminus \{(\emptyset, x^\emptyset), (N, x^*)\}$. Note that $S \neq \emptyset$. Consider the case where $x_0 < 2(W(2) - L(0)) = x_0^*$. In this case, $(S, x) \to_{\{0\}} (\emptyset, x^\emptyset) \to_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \to_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (S, x)$ by $x_0^* > x_0$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$. Assume, therefore, that $x_0 \geq 2(W(2) - L(0)) = x_0^*$ hereafter. Consider the case where $S = \{i\}$ for some $i = 1, 2$. In this case,

$$x_i = W(1) - x_0 \leq W(1) - 2(W(2) - L(0)) < L(0) - \alpha \leq x_i^*.$$

Thus, $(S, x) \to_{\{i\}} (\emptyset, x^\emptyset) \to_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \to_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (S, x)$ by $x_i < x_i^*$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$.

Assume additionally that $S = N$ hereafter. If $x_1 < L(0) + \alpha = x_1^*$, then $(N, x) \to_{\{1\}} (\emptyset, x^\emptyset) \to_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \to_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (N, x)$ by $x_1^* > x_1$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$. Therefore, assume further that $x_1 \geq L(0) + \alpha = x_1^*$ hereafter. By $x \neq x^*$, either $x_0 > x_0^*$ or $x_1 > x_1^*$. Then, $x_2 < x_2^* = L(0) - \alpha$. Then, $(N, x) \to_{\{2\}} (\emptyset, x^\emptyset) \to_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \to_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (S, x)$ by $x_2^* > x_2$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$. Hence, $\{(N, x^*)\}$ is externally stable.

Therefore, the union of singleton farsighted stable sets is possibly strictly larger than $\mathring{C}$. Moreover, the farsighted stable set in Example 1 is not included even in the core. This is a departure from the results of the literature on close relationships between single-payoff farsighted stable sets and cores. Mauleon, et al. (2011) showed that a set of matchings is a farsighted stable set if and only if it is a singleton consisting of a core matching (a strong core matching, respectively) in one-to-one matching (many-to-one matching with substitutability, respectively). Ray and Vohra (2015a) characterized single-payoff farsighted stable sets by separable payoffs that is close to the core since any separable payoff vector is in the core and any payoff vector in the relative interior of the core is separable. Applying this result, they also showed that a single-payoff set is a farsighted stable set if and only if it yields a payoff in the relative interior of the core in a simple game called oligarchic where the set of veto players is a winning coalition. Chander (2015) considered farsighted stable sets in partition function games, and he showed that single-payoff farsighted stable sets are characterized by the strong core. Hirai (2018) obtained a similar characterization as these results in a certain class of strategic games called strategic games with dominant punishment strategies. He showed that a set of strategy profiles yielding a single payoff vector is a farsighted stable set if and only if the set satisfies a condition called inclusiveness, where an inclusive set has a close relationship with the strict α-core. To recover such a relationship between singleton farsighted stable sets and the core, we need to restrict our attention to symmetric farsighted stable sets.

Proposition 3 *Let* $(S,x) \in \bar{X}$*. Then,* $\{(S,x)\}$ *is a farsighted stable set if and only if* $(S,x) \in \mathring{C}$*.*

Proof. The sufficiency follows from Proposition 2. Thus, we show the necessity.

Fix an arbitrary $(S,x) \in \bar{X}$. Assume that $\{(S,x)\}$ is a farsighted stable set. Suppose that $(S,x) \notin \mathring{C}$. Then, there exist $(T,y) \in X \setminus \{(S,x)\}$ and $P \subseteq \{0\} \cup N$ such that $(S,x) \to_P (T,y)$ and $y_i \geq x_i$ for all $i \in P$. Since $\{(S,x)\}$ is a farsighted stable set, $(S,x) \succ (T,y)$. Thus, there exist sequences of outcomes $(Q^0,z^0),...,(Q^m,z^m)$ and coalitions $R^1,...,R^m$ such that $(Q^0,z^0) = (T,y)$, $(Q^m,z^m) = (S,x)$, and for all $h = 1,...,m$, $(Q^{h-1},z^{h-1}) \to_{R^h} (Q^h,z^h)$ and $z_i^{h-1} < x_i$ for all $i \in R^h$.

First, consider the case where $T \neq \emptyset$. Then, $P = \{0\} \cup T$ by Assumption 2(ii). By $y_i \geq x_i$ for all $i \in \{0\} \cup T$, $R^1 \cap (\{0\} \cup T) = \emptyset$. This contradicts Assumption 2.

Next, consider the case where $T = \emptyset$. Note that $(T,y) = (\emptyset, x^\emptyset)$. By $(S,x) \neq (T,y)$, $S \neq \emptyset$. Then, $P \subseteq \{0\} \cup S$ by Assumption 2(i). Assume that $0 \in P$. Then, $x_0^\emptyset = 0 = y_0 \geq x_0$. Thus, $(Q^1,z^1) = (\emptyset, x^\emptyset)$. For each $h = 2,...,m$, if $(Q^{h-1},z^{h-1}) = (\emptyset, x^\emptyset)$, then $0 \notin R^h$ and $(Q^h,z^h) = (\emptyset, x^\emptyset)$ by $x_0^\emptyset \geq x_0$. Thus, $(Q^m,z^m) = (\emptyset, x^\emptyset)$. This contradicts the nonemptiness of S. Assume therefore that $0 \notin P$. Thus, $P \subseteq S$. By the symmetry of (S,x) and $L(0) = y_i \geq x_i$ for all $i \in P$, we have that $x_i^\emptyset = L(0) \geq x_i$ for all $i \in S$. Moreover, $x_i^\emptyset = L(0) > L(s) \geq x_i$ for all $i \in N \setminus S$. Thus, $R^1 = \{0\}$. It follows that $(Q^1,z^1) = (\emptyset, x^\emptyset)$. For each $h = 2,...,m$, if $(Q^{h-1},z^{h-1}) = (\emptyset, x^\emptyset)$, then $R^h = \{0\}$ and $(Q^h,z^h) = (\emptyset, x^\emptyset)$ by $x_i^\emptyset = L(0) \geq x_i$ for all $i \in N$. Thus, $(Q^m,z^m) = (\emptyset, x^\emptyset)$. This contradicts the nonemptiness of S. Hence, $(S,x) \in \mathring{C}$. ■

4.3 Comparison with the main simple set: An example

A symmetric farsighted stable set is not a (myopic) stable set defined by von Neumann and Morgenstern, as noted in the Introduction. It seems more closely related to the main simple set in a simple game as long as $B(\alpha)$ is a singleton, although our model is not a simple game.[k] Ray and Vohra (2015a,b) showed that a main simple set is a farsighted stable set if it is a (myopic) von Neumann-Morgenstern stable set.[l,m] In our model, $\bar{X}(\alpha)$ becomes a farsighted stable set, even though it is not a von Neumann-Morgensterns stable set. Moreover, as shown in Theorem 1, $\bar{X}(\alpha)$ is a farsighted stable set even if $B(\alpha)$ is not a singleton. These facts are illustrated with Example 2 and depicted in Figure 1.

Example 2 Let $N = \{0, 1, 2\}$, $W(1) = 100$, $W(2) = 50$, $L(0) = 30$, $L(1) = 0$. Then, we can construct a coalitional patent licensing game with transferable utility according to Watanabe and Muto (2008); $v(N) = 2W(2) = 100$, $v(\{0, 1\}) = v(\{0, 2\}) = W(1) = 100$, $v(\{1, 2\}) = 2L(0) = 60$, $v(\{1\}) = v(\{2\}) = L(1) = 0$, $v(\{0\}) = 0$, and $v(\emptyset) = 0$. Note that $v(\{1, 2\}) > 0$ is the main departure from a simple game. Note also that not only the relative interior of the core but also the core itself is empty in this example, while the core becomes nonempty when $v(\{1, 2\})$ is changed to 0, because the game becomes a simple game. In Figure 1, the symmetric farsighted stable set $\bar{K} = \bar{X}(\alpha)$ is depicted for the case of $\alpha = 20$ as well as a main simple set for the patent licensing game with transferable utility where $\{0, 1\}$ and $\{0, 2\}$ are treated as minimal winning coalitions. This shows the similarity between our symmetric farsighted stable set and main simple set. However, it can be easily confirmed that $\{(50, 50, 0), (50, 0, 50)\}$ is not a von Neumann-Morgenstern stable set because it does not directly dominate $(0, 50, 50)$.

[k] A simple game is a coalitional game with transferable utility (N, v) associated with a set of winning coalitions $\mathcal{W}$. The characteristic function v is defined as $v(S) = 1$ if $S \in \mathcal{W}$, while $v(S) = 0$ if $S \notin \mathcal{W}$. A winning coalition S is said to be minimal if there is no $T \in \mathcal{W}$ with $T \subsetneq S$. A main simple set associated with a vector $m \in \mathbb{R}^n$ satisfying $\sum_{i \in S} m_i = 1$ for any minimal winning coalition S is defined as $\{x^S | S \text{ is a minimal winning coalition}\}$, where x^S is an imputation such that $x_i^S = m_i$ for all $i \in S$ and $x_i^S = 0$ for all $i \notin S$.

[l] In a coalitional game with transferable utility (N, v), an imputation is said to directly dominate another imputation y if there exists a nonempty $S \subseteq N$, $\sum_{i \in S} x_i \leq v(S)$ and $x_i > y_i$ for all $i \in S$. In (N, v), a set of imputations K is a von Neumann-Morgenstern stable set if for any $x, y \in K$, x does not directly dominate y (internal stability), and for any imputation $x \notin K$, some $y \in K$ directly dominates x (external stability).

[m] The phrase "main simple set" is cited from Ray and Vohra (2019a). When a main simple set is a von Neumann-Morgenstern stable set, it is what von Neumann and Morgenstern (1944) called the main simple solution.

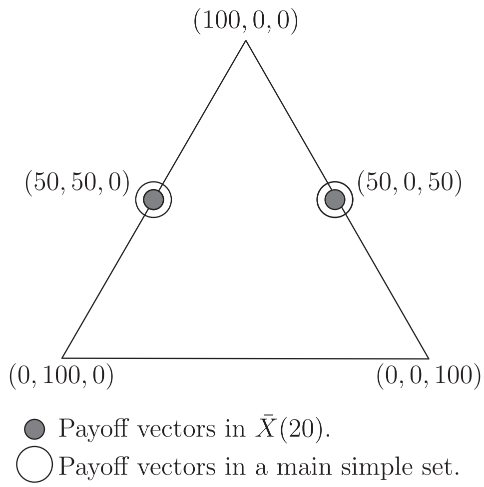

Figure 1: A farsighted stable set and the main simple solution.

5 Maximality and Rational Expectation

In paths constituting an indirect dominance relation, any coalition appeared in the paths is eventually made better off. Nevertheless, a coalition, possibly not appeared in the paths, may make its members even better by intervening the original paths. In this sense, an indirect dominance relation is sometimes incredible. Dutta and Vartiainen (2017) and Ray and Vohra (2019a) recently considered such a problem and proposed conditions that require an indirect dominance relation be robust against such a coalitional intervention, respectively. Dutta and Vartiainen (2017) defined a farsighted stable set called a history dependent strongly rational expectation farsighted stable set with indirect dominance relation under some restriction. Ray and Vohra (2019a) defined an absolutely maximal farsighted stable set as a farsighted stable set whose external stability is satisfied for indirect dominance relation under another restriction. In this section, we examine whether our farsighted stable sets described in Section 4 are also absolutely maximal and/or history dependent strongly rational expectation. Dutta and Vartiainen (2017) and Ray and Vohra (2019a) use different terms for the same definitions for some concepts. This paper basically follows the terminology of Ray and Vohra (2019a).

We begin with preparation for defining these farsighted stable sets. A history h is represented by a finite sequence of outcomes and (possibly empty) sets of players $((S^0, x^0), (T^1, (S^1, x^1)), ..., (T^m, (S^m, x^m)))$ such that for all $\ell = 1, ..., m$,

- if $(S^{\ell-1}, x^{\ell-1}) \neq (S^\ell, x^\ell)$, then $(S^{\ell-1}, x^{\ell-1}) \rightarrow_{T^\ell} (S^\ell, x^\ell)$, where $T^\ell \neq \emptyset$,

- if $(S^{\ell-1}, x^{\ell-1}) = (S^\ell, x^\ell)$, then $T^\ell = \emptyset$.

A history consisting of a single outcome is called an initial history. Note that an initial history may be consisting of an outcome where some firms have already been licensed. This means that a negotiation possibly begins with a situation where the patent holder and some firms have been signing a contract. For any history h, let $(S(h), x(h))$ denote the last outcome of h. Formally, $(S(h), x(h)) = (S^m, x^m)$ when $h = ((S^0, x^0), (T^1, (S^1, x^1)), ..., (T^m, (S^m, x^m)))$.

We here describe a concept of the negotiation process that prescribes what an outcome follows a history and who induces it. Note that a negotiation process depends not only on a current outcome, but also the whole history. Given a history h, a negotiation process is a mapping σ that assigns a pair of a (possibly empty) set of players and an outcome to any history that is inducible from the last outcome of the history h via the assigned players. For any history h, denote $\sigma(h) = (R(h), (Q(h), z(h)))$ where $(S(h), x(h)) \rightarrow_{R(h)} (Q(h), z(h))$ if $(S(h), x(h)) \neq (Q(h), z(h))$, while $R(h) = \emptyset$ if $(S(h), x(h)) = (Q(h), z(h))$. We further define $\sigma^k(h)$ for any $k \geq 1$ inductively. For any history h, define $\sigma^1(h) = \sigma(h)$ and $\sigma^k(h) = \sigma(h, \sigma^1(h), ..., \sigma^{k-1}(h))$ for any $k > 1$. Equivalently, $\sigma^1(h) = (R(h), (Q(h), z(h)))$ and $\sigma^k(h) = (R^k(h), (Q^k(h), z^k(h)))$ such that

- $R^k(h) = R(h, \sigma^1(h), ..., \sigma^{k-1}(h))$ and
- $(Q^k(h), z^k(h)) = (Q(h, \sigma^1(h), ..., \sigma^{k-1}(h)), z(h, \sigma^1(h), ..., \sigma^{k-1}(h)))$

for any $k > 1$.

An outcome (S, x) is absorbing under a negotiation process σ if for any history h with $(S(h), x(h)) = (S, x)$, $(Q(h), z(h)) = (S(h), x(h))$.[n] A negotiation process σ is called an absorbing process if for any history h, there exist some absorbing outcome (S, x) and $k > 0$ such that $(Q^{k'}(h), z^{k'}(h)) = (S, x)$ for any $k' \geq k$. For any absorbing process σ, let $(S^\sigma(h), x^\sigma(h))$ be the absorbing outcome reached from h. For any absorbing process σ and a history h such that $(S(h), x(h))$ is not an absorbing outcome, we say that σ yields $(S^\sigma(h), x^\sigma(h)) \succ (S(h), x(h))$ from h if

- $(Q^{\ell-1}(h), z^{\ell-1}(h)) \rightarrow_{R^\ell} (Q^\ell(h), z^\ell(h))$ for all $\ell = 1, ..., k$, where $(Q^0(h), z^0(h)) = (S(h), x(h))$, $(Q^k(h), z^k(h)) = (S^\sigma(h), x^\sigma(h))$, and $(Q^\ell(h), z^\ell(h)) \neq (S^\sigma(h), x^\sigma(h))$ for all $\ell = 1, ..., k-1$;

[n] An absorbing outcome is referred as a stationary outcome in Dutta and Vartiainen (2017).

- $x_i^\sigma(h) > z_i^{\ell-1}(h)$ for all $i \in R^\ell$,

where k is the minimum integer such that $(Q^k(h), z^k(h)) = (S^\sigma(h), x^\sigma(h))$.

An absorbing process σ is coalitionally acceptable if for each history h, if $R(h) \neq \emptyset$, then $x_i^\sigma(h) \geq x_i(h)$ for all $i \in R(h)$. An absorbing process σ is absolutely maximal if for any history h, there exist no coalition T^+ and outcome (S^+, x^+) with $(S(h), x(h)) \rightarrow_{T^+} (S^+, x^+)$ such that $x_i^\sigma(h^+) > x_i^\sigma(h)$ for all $i \in T^+$, where $h^+ = (h, (T^+, (S^+, x^+)))$.

An absorbing process guarantees that a negotiation process eventually reaches to some absorbing outcome. Coalitional acceptability requires that no player in coalitions appeared along a negotiation process have no incentive to deviate from a prescribed process. Absolute maximality requires that no coalition have an incentive to intervene a prescribed process. Therefore, an indirect dominance relation would be credible if it obeyed the way of negotiation prescribed by an absolutely maximal process. Ray and Vohra (2019a) defined a farsighted stable set absolutely maximal as follows.

Definition 4 (Ray and Vohra, 2019) *A farsighted stable set K is absolutely maximal if there exists an absorbing, coalitionally acceptable, and absolutely maximal process σ such that*

- *K is the set of all absorbing outcomes of σ;*
- *for any initial history $h = (S^0, x^0)$ with $(S^0, x^0) \notin K$, σ yields $(S^\sigma(h), x^\sigma(h)) \succ (S^0, x^0)$ from h;*
- *for any history $h = ((S^0, x^0), (T^1, (S^1, x^1)))$ with $(S^0, x^0) \in K$ and $(S^1, x^1) \notin K$, σ yields $(S^\sigma(h), x^\sigma(h)) \succ (S^1, x^1)$ from h.*

Dutta and Vartiainen (2017) defined an indirect dominance relation with rational expectations. An absorbing process σ is said to be a history dependent strongly rational (HSR) process if it satisfies the following (I), (E), and (M*).

(I) For any history h with $(S(h), x(h)) = (Q(h), z(h))$, there exist no $(S', x') \in X$ and $T \subseteq \{0\} \cup N$ such that $(S(h), x(h)) \rightarrow_T (S', x')$ and $x_i^\sigma(h') > x_i(h)$ for all $i \in T$, where $h' = (h, (T', (S', x')))$.

(E) For any history h with $(S(h), x(h)) \neq (Q(h), z(h))$, σ yields an indirect dominance relation $(S^\sigma(h), x^\sigma(h)) \succ (S(h), x(h))$ from h.

(M*) For any history h with $(S(h), x(h)) \neq (Q(h), z(h))$, there exist no $(S^+, x^+) \in X$ and nonempty $T^+ \subseteq \{0\} \cup N$ such that $R(h) \cap T^+ \neq \emptyset$, $(S(h), x(h)) \rightarrow_{T^+} (S^+, x^+)$, and $x_i^\sigma(h^+) > x_i^\sigma(h)$ for all $i \in T^+$, where $h^+ = (h, (T^+, (S^+, x^+)))$.

Definition 5 (Dutta and Vartiainen, 2017) *A set of outcomes K is a history dependent strongly rational expectation farsighted stable set (HSREFS) if there exists a HSR process σ such that K is the set of absorbing outcomes of σ.*

Conditions (I) and (E) of a HSR process σ requires that the set of absorbing outcomes satisfy internal and external stabilities under σ, respectively. Condition (M*) of a HSR process σ requires that an indirect dominance relation constructed from σ is robust to an intervention by a coalition.[o] Note that this coalition must have a nonempty intersection with a coalition prescribed by σ. This is the departure from the absolute maximality of σ. Note that a HSREFS may not be a farsighted stable set because an indirect dominance relation must be yielded from a given negotiation process for both internal and external stabilities as also pointed out by Ray and Vohra (2019a). Though, we will show that our farsighted stable set described in Section 4 is a HSREFS.

Remark 1 Let σ be an absorbing process. Then, the following facts are obvious.

- If σ is absolutely maximal, then σ satisfies (M*).
- If σ satisfies (E), then σ satisfies coalitional acceptability.

Ray and Vohra (2019a) showed that a farsighted stable set K is absolutely maximal if K satisfies two conditions called Property A and Property B. Although our model is slightly departed from their formulation by the presence of a negative externality of a patented technology, Property A can be straightforwardly embedded into our model as follows.[p]

Property A*: Assume that there exist distinct $(S, x), (T, y) \in K$ and $j \in \{0\} \cup N$ such that $x_j > y_j$. Then, there exists some $(Q, z) \in K$ such that $z_j \leq y_j$ and $z_i \geq x_i$ for all $i \neq j$.

[o] Dutta and Vartiainen (2017) also proposed a history dependent rational expectation farsighted stable set (HREFS), which is defined by an indirect dominance relation satisfying conditions (I), (E), and a weaker version of (M*).

[p] Property B follows from internal stability in our model when it is appropriately embedded. Therefore, we omit it.

Ray and Vohra (2019b) showed that a main simple set satisfies Property A in a class of symmetric simple game. As we discussed at the end of Section 4, our farsighted stable set characterized by Theorem 1 has a similarity with a main simple set. However, there are two difficulties to apply the result of Ray and Vohra (2019a): a coalitional game derived from our model is departed from a symmetric simple game as shown in Example 2, and our symmetric farsighted stable set characterized by Theorem 1 may not satisfy Condition A*, though it can be easily confirmed that Property A* is satisfied when $B(\alpha)$ is a singleton for a given $\alpha \in A$. The latter is shown by the following example.

Example 3 Let $N = \{1, 2, 3, 4\}$ and the profit functions W and L are given as follows.

s	0	1	2	3	4
$W(s)$	-	17	14	12	11
$L(s)$	10	9	8	7	-

Let $\alpha = 1$. Then, $B(1) = \{1, 2\}$, and

$$\bar{X}(1) = \{(\{k\}, x^{\{k\}}) \in \bar{X} |\ k \in N,\ x_0^{\{k\}} = 6,\ x_k^{\{k\}} = 11,\ x_j^{\{k\}} = 9 \text{ for all } j \neq k\} \\ \cup \{(S, x^S) \in \bar{X} |\ |S| = 2, x_0^S = 6,\ x_j^S = 11 \text{ for all } j \in S,\ x_j^S = 8 \text{ for all } j \notin S\}$$

is a symmetric farsighted stable set by Theorem 1. Consider two outcomes $(S, x), (T, y) \in \bar{X}(1)$ such that $(S, x) = (\{1\}, 6, 11, 9, 9, 9)$ and $(T, y) = (\{1, 2\}, 6, 11, 11, 8, 8)$. By $x_3 = 9 > 8 = y_3$, if $\bar{X}(1)$ satisfies Property A*, then there exists some $(Q, z) \in \bar{X}(1)$ such that $z_3 \leq y_3$ and $z_i \geq x_i$ for all $i = 0, 1, 2, 4$. By $z_3 \leq y_3 = 8$, $|Q| = 2$ and $3 \notin Q$. Then, there exists some $h \in \{1, 2, 4\}$ such that $h \notin Q$. Hence, $z_h = 8 < x_h$. Thus, our symmetric farsighted stable set may not satisfy Property A* in our abstract game model.

In spite of such difficulties, we hereby prove the absolute maximality in any symmetric farsighted stable set characterized in Section 4 by constructing an absorbing, coalitionally acceptable, and absolutely maximal process. Dutta and Vartiainen (2017) showed a necessary and sufficient condition for a set of outcomes to be HSREFS in an abstract game. Therefore, their result apply to our model. However, we also prove that our farsighted stable sets characterized in Section 4 by ourselves because it requires us to introduce an additional definition and can be proved in a unified way with the absolute maximality in the farsighted stable sets.

Theorem 2 *For any $\alpha \in A$, $\bar{X}(\alpha)$ is an absolutely maximal farsighted stable set and a HSREFS.*

Proof. We begin with some preparations. Fix an arbitrary $\alpha \in A$. Fix an arbitrary $s^* \in B(\alpha)$. Note that if $(S, x) \notin \bar{X}(\alpha)$, then $x_0 < s^*(W(s^*) - L(0) - \alpha)$ or $x_i < L(0) + \alpha$ for some $i \in S$. For each $i \in N$, define $\bar{S}(i) = \{i, i+1, ..., i+s^*-1\}$, where $n+k \equiv k$ for all $k = 1, ..., n$. For any nonempty $S \subseteq N$, define $y^S \in \mathbb{R}^{n+1}$ a payoff vector such that

$$y_i^S = \begin{cases} s(W(s) - L(0) - \alpha) & \text{if } i = 0; \\ L(0) + \alpha & \text{if } i \in S; \\ L(s) & \text{if } i \in N \setminus S. \end{cases}$$

Therefore, $(S, y^S) \in \bar{X}(\alpha)$ when $s \in B(\alpha)$.

Define a negotiation process $\bar{\sigma}$ for each history h as follows. The following conditions (A)–(C) will be referred throughout this proof.

(A) Let h be a history such that outcomes in h are all null outcomes. In this case, define $\bar{\sigma}(h) = (\{0\} \cup \bar{S}(1), (\bar{S}(1), y^{\bar{S}(1)}))$.

(B) Let h be a history such that $(S(h), x(h)) = (\emptyset, x^{\emptyset})$ and h includes at least one non-null outcome. Denote (S', x') be a non-null outcome that appears at last in h except for the null outcome.

  (i) If $x'_0 < s^*(W(s^*) - L(0) - \alpha)$, then define $\bar{\sigma}(h) = (\{0\} \cup \bar{S}(1), (\bar{S}(1), y^{\bar{S}(1)}))$.

  (ii) If $x'_0 \geq s^*(W(s^*) - L(0) - \alpha)$ and $x'_i < L(0) + \alpha$ for some $i \in S'$, then define $\bar{\sigma}(h) = (\{0\} \cup \bar{S}(j'), (\bar{S}(j'), y^{\bar{S}(j')}))$ where $j' = \min\{i \in S' | x'_i < L(0) + \alpha\}$.

  (iii) If $(S', x') \in \bar{X}(\alpha)$, then define $\bar{\sigma}(h) = (\{0\} \cup S', (S', x'))$.

(C) Let h be a history such that $(S(h), x(h)) \neq (\emptyset, x^{\emptyset})$.

  (i) If $x_0(h) < s^*(W(s^*) - L(0) - \alpha)$, then define $\bar{\sigma}(h) = (\{0\}, (\emptyset, x^{\emptyset}))$.

  (ii) If $x_0(h) \geq s^*(W(s^*) - L(0) - \alpha)$ and $x_i(h) < L(0) + \alpha$ for some $i \in S(h)$, then define $\bar{\sigma}(h) = (\{j'\}, (\emptyset, x^{\emptyset}))$ where $j' = \min\{i \in S(h) | x_i(h) < L(0) + \alpha\}$.

  (iii) If $(S(h), x(h)) \in \bar{X}(\alpha)$, then define $\bar{\sigma}(h) = (\emptyset, (S(h), x(h)))$.

By the definition of $\bar{\sigma}$, for any history h, $(S(h), x(h)) = (Q(h), z(h))$ if and only if $(S(h), x(h)) \in \bar{X}(\alpha)$, which happens in (C)-(iii). Thus, $\bar{X}(\alpha)$ is the set of absorbing outcomes. By this fact, $\bar{\sigma}$ satisfies the second and third bullets in the definition of an absolutely maximal farsighted stable set (Definition 4) if $\bar{\sigma}$ satisfies condition (E).

Therefore, it suffices to show that $\bar{\sigma}$ satisfies conditions (I) and (E), and $\bar{\sigma}$ is absorbing and absolute maximal by Remark 1.

We specify which outcome is eventually reached from a given history under $\bar{\sigma}$. Let h be a history. First, assume that h satisfies (A) or (B). Thus, $(S(h), x(h)) = (\emptyset, x^{\emptyset})$. Then, there exists some $(\bar{S}, y^{\bar{S}}) \in \bar{X}(\alpha)$ such that $\bar{\sigma}(h) = (\{0\} \cup \bar{S}, (\bar{S}, y^{\bar{S}}))$ by (A) and (B) in the definition of $\bar{\sigma}$, and

$$\bar{\sigma}^k(h) = (\emptyset, (\bar{S}, y^{\bar{S}})) \text{ for any } k \geq 2 \tag{4}$$

by (C)-(iii) in the definition of $\bar{\sigma}$. Note that $(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) = (\bar{S}, y^{\bar{S}})$ if $\bar{\sigma}$ is proved to be an absorbing process. Moreover, $(\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, y^{\bar{S}})$, $y_0^{\bar{S}} = s^*(W(s^*) - L(0) - \alpha) > 0 = x_0^{\emptyset}$, and $y_i^{\bar{S}} = L(0) + \alpha > L(0) = x_i^{\emptyset}$ for all $i \in \bar{S}$. Therefore, $(\bar{S}, y^{\bar{S}}) \succ (\emptyset, x^{\emptyset})$ that is equivalent to

$$(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) \succ (S(h), x(h)), \tag{5}$$

if $\bar{\sigma}$ is proved to be an absorbing process.

Next, assume that h satisfies (C). We distinguish three cases.

Case (i) h satisfies (C)-(i).

By the definition of $\bar{\sigma}$, $\bar{\sigma}(h) = (\{0\}, (\emptyset, x^{\emptyset}))$ follows after h. Then, $(h, \bar{\sigma}(h))$ satisfies (B)-(i). Thus, $\bar{\sigma}^2(h) = (\{0\} \cup \bar{S}(1), (\bar{S}(1), y^{\bar{S}(1)}))$, and

$$\bar{\sigma}^k(h) = (\emptyset, (\bar{S}(1), y^{\bar{S}(1)})) \text{ for any } k \geq 3 \tag{6}$$

by (C)-(iii) in the definition of $\bar{\sigma}$, which is an absorbing outcome. Note that $(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) = (\bar{S}(1), y^{\bar{S}(1)})$ if $\bar{\sigma}$ is proved to be an absorbing process. Moreover, $(S(h), x(h)) \rightarrow_{\{0\}} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup \bar{S}(1)} (\bar{S}(1), y^{\bar{S}(1)})$ and

$$\begin{aligned} y_0^{\bar{S}(1)} &= s^*(W(s^*) - L(0) - \alpha) > x_0(h); \\ y_0^{\bar{S}(1)} &= s^*(W(s^*) - L(0) - \alpha) > 0 = x_0^{\emptyset}; \\ y_i^{\bar{S}(1)} &= L(0) + \alpha > L(0) = x_i^{\emptyset} \text{ for all } i \in \bar{S}(1). \end{aligned}$$

Therefore, $(\bar{S}(1), y^{\bar{S}(1)}) \succ (S(h), x(h))$ that is equivalent to

$$(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) \succ (S(h), x(h)) \tag{7}$$

if $\bar{\sigma}$ is proved to be an absorbing process.

Case (ii) h satisfies (C)-(ii).

By the definition of $\bar{\sigma}$, $\bar{\sigma}(h) = (\{j'\}, (\emptyset, x^{\emptyset}))$ follows after h, where $j' = \min\{i \in S(h) | x_i(h) < L(0) + \alpha\}$. Then, $(h, \bar{\sigma}(h))$ satisfies (B)-(ii). Thus, $\bar{\sigma}^2(h) = (\{0\} \cup \bar{S}(j'), (\bar{S}(j'), y^{\bar{S}(j')}))$ and

$$\bar{\sigma}^k(h) = (\emptyset, (\bar{S}(j'), y^{\bar{S}(j')})) \text{ for any } k \geq 3 \tag{8}$$

by (C)-(iii) in the definition of $\bar{\sigma}$. Note that $(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) = (\bar{S}(j'), y^{\bar{S}(j')})$ if $\bar{\sigma}$ is proved to be an absorbing process. Moreover, $(S(h), x(h)) \rightarrow_{\{j'\}} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup \bar{S}(j')} (\bar{S}(j'), y^{\bar{S}(j')})$ and

$$\begin{aligned} y_{j'}^{\bar{S}(j')} &= L(0) + \alpha > x_{j'}(h); \\ y_0^{\bar{S}(j')} &= s^*(W(s^*) - L(0) - \alpha) > 0 = x_0^{\emptyset}; \\ y_i^{\bar{S}(j')} &= L(0) + \alpha > L(0) = x_i^{\emptyset} \text{ for all } i \in \bar{S}(j'). \end{aligned}$$

Therefore, $(\bar{S}(j'), y^{\bar{S}(j')}) \succ (S(h), x(h))$ that is equivalent to

$$(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) \succ (S(h), x(h)) \tag{9}$$

if $\bar{\sigma}$ is proved to be an absorbing process.

Case (iii) h satisfies (C)-(iii).

By the definition of $\bar{\sigma}$,

$$\bar{\sigma}^k(h) = (\emptyset, (S(h), x(h))) \text{ for any } k \geq 1. \tag{10}$$

Note that $(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) = (S(h), x(h))$ if $\bar{\sigma}$ is proved to be and absorbing process.

By (4), (6), (8), and (10), $\bar{\sigma}$ is an absorbing process. Thus, $(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h))$ is well-defined for any history h. By (5), (7), and (9), $\bar{\sigma}$ satisfies condition (E).

We show that $\bar{\sigma}$ satisfies (I). Let $\hat{h}$ be a history such that $(S(\hat{h}), x(\hat{h})) = (Q(\hat{h}), z(\hat{h}))$. Then, $(S(\hat{h}), x(\hat{h})) \in \bar{X}(\alpha)$. Suppose that there exist some $(S', x') \in X$ and nonempty $T \subseteq \{0\} \cup N$ such that $(S(\hat{h}), x(\hat{h})) \rightarrow_{T'} (S', x')$ and $x_i^{\bar{\sigma}}(h') > x_i(\hat{h})$ for all $i \in T'$, where $h' = (\hat{h}, (T', (S', x')))$. Then, $T' \subseteq N$ because $x_0^{\bar{\sigma}}(h') = x_0(\hat{h}) = s^*(W(s^*) - L(0) - \alpha)$. Thus, $T' \subseteq S(\hat{h})$ and $(S', x') = (\emptyset, x^{\emptyset})$ by Assumption 2(i). However, this contradicts that $x_i^{\bar{\sigma}}(h') > x_i(\hat{h})$ for all $i \in T'$ because $x_i(\hat{h}) = L(0) + \alpha$ for all $i \in T'$ and the payoff of a firm is at most $L(0) + \alpha$ at any absorbing outcome. Hence, $\bar{\sigma}$ satisfies (I).

It remains to show that $\bar{\sigma}$ is an absolutely maximal process. Suppose that there exist a coalition T^+ and outcome (S^+, x^+) such that $(S(h), x(h)) \rightarrow_{T^+} (S^+, x^+)$ and

$x_i^{\bar{\sigma}}(h^+) > x_i^{\bar{\sigma}}(h)$ for all $i \in T^+$, where $h^+ = (h, (T^+, (S^+, x^+)))$. Then, $T^+ \subseteq N$ because the payoff of the patent holder is $s^*(W(s^*) - L(0) - \alpha)$ at any absorbing outcome. Thus, $(S^+, x^+) = (\emptyset, x^\emptyset)$ by $(S(h), x(h)) \rightarrow_{T^+} (S^+, x^+)$ and Assumption 2(i). By Assumption 2, $(S(h), x(h)) \neq (\emptyset, x^\emptyset)$ because $(\emptyset, x^\emptyset) \rightarrow_T (\emptyset, x^\emptyset)$ is prohibited for any coalition T. By $(S(h), x(h)) \neq (\emptyset, x^\emptyset)$ and $(S^+, x^+) = (\emptyset, x^\emptyset)$, h satisfies (C) and h^+ satisfies (B) in the definition of $\bar{\sigma}$. Denote $h^* = (h, \bar{\sigma}(h))$.

First, assume that h satisfies (C)-(i). Then, both $h^+ = (h, (T^+, (\emptyset, x^\emptyset)))$ and $h^* = (h, (\{0\}, (\emptyset, x^\emptyset)))$ satisfy (B)-(i). By the definition of $\bar{\sigma}$, $\bar{\sigma}(h^+) = \bar{\sigma}(h^*) = (\{0\} \cup \bar{S}(1), (\bar{S}(1), y^{\bar{S}(1)}))$, and thus, $\bar{\sigma}^k(h^+) = \bar{\sigma}^k(h^*) = (\emptyset, (\bar{S}(1), y^{\bar{S}(1)}))$ for any $k \geq 2$ by (C)-(iii) in the definition of $\bar{\sigma}$. Thus, $(\bar{S}(1), y^{\bar{S}(1)}) = (S^{\bar{\sigma}}(h^+), x^{\bar{\sigma}}(h^+)) = (S^{\bar{\sigma}}(h^*), x^{\bar{\sigma}}(h^*)) = (S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h))$. This contradicts that $x_i^{\bar{\sigma}}(h^+) > x_i^{\bar{\sigma}}(h)$ for all $i \in T^+$.

Next, assume that h satisfies (C)-(ii). Then, both $h^+ = (h, (T^+, (\emptyset, x^\emptyset)))$ and $h^* = (h, (\{j'\}, (\emptyset, x^\emptyset)))$ satisfy (B)-(ii), where $j' = \min\{i \in S(h) | x_i(h) < L(0) + \alpha\}$. By the definition of $\bar{\sigma}$, $\bar{\sigma}(h^+) = \bar{\sigma}(h^*) = (\{0\} \cup \bar{S}(j'), (\bar{S}(j'), y^{\bar{S}(j')}))$, and thus, $\bar{\sigma}^k(h^+) = \bar{\sigma}^k(h^*) = (\emptyset, (\bar{S}(j'), y^{\bar{S}(j')}))$ for any $k \geq 2$ by (C)-(iii) in the definition of $\bar{\sigma}$. Thus, $(\bar{S}(j'), y^{\bar{S}(j')}) = (S^{\bar{\sigma}}(h^+), x^{\bar{\sigma}}(h^+)) = (S^{\bar{\sigma}}(h^*), x^{\bar{\sigma}}(h^*)) = (S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h))$. This contradicts that $x_i^{\bar{\sigma}}(h^+) > x_i^{\bar{\sigma}}(h)$ for all $i \in T^+$.

Finally assume that h satisfies (C)-(iii). Then, $\bar{\sigma}^k(h) = (\emptyset, (S(h), x(h)))$ for any $k \geq 1$. Thus, $(S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h)) = (S(h), x(h))$. By (B)-(iii) in the definition of $\bar{\sigma}$, $\bar{\sigma}(h^+) = (\{0\} \cup S(h), (S(h), x(h)))$ and $\bar{\sigma}^k(h^+) = (\emptyset, (S(h), x(h)))$ for any $k \geq 2$. Thus, $(S^{\bar{\sigma}}(h^+), x^{\bar{\sigma}}(h^+)) = (S(h), x(h)) = (S^{\bar{\sigma}}(h), x^{\bar{\sigma}}(h))$. This contradicts that $x_i^{\bar{\sigma}}(h^+) > x_i^{\bar{\sigma}}(h)$ for all $i \in T^+$.

Every case yields a contradiction. Hence, $\bar{\sigma}$ is an absolutely maximal process. This completes the proof. ■

6 Concluding remarks

As a main result, this paper characterized symmetric farsighted stable set of an abstract game formalized for patent licensing negotiations as a set of outcomes where the patent holder maximizes its own profit, provided that each licensee firm is allowed to enjoy a given net profit which is regarded as an established order of society. We also show the equivalence between singleton farsighted stable sets and the relative interior of the core is obtained if we restrict our attention to symmetric farsighted stable sets. Further, we show that the symmetric farsighted stable sets are also the absolutely maximal farsighted stable sets and with the history dependent strongly rational expectation farsighted stable set.[q] Those relationships imply that the main result stated above is strong and robust.

The patent holder's profit is a main concern in the literature on patent licensing, as noted in Section 1. It can be easily derived that the supremum of a patent holder's profit supported by symmetric farsighted stable sets is $\max_{s=1,\ldots,n} s(W(s) - L(0))$. The supremum is thus consistent with the results for bargaining set (Watanabe and Muto, 2008) and kernel and nucleolus (Kishimoto and Watanabe, 2017). The infimum is, however, 0, while the patent holder obtains more when solutions mentioned above and vNM stable set analyzed by Hirai and Watanabe (2018) are applied to the model of Watanabe and Muto (2008).

In the model of Watanabe and Muto (2008), each firm is supposed to have a pessimistic anticipation that other firms are licensed in such a way that its gross profit is minimized when it breaks off the license negotiations. This pessimistic viewpoint which inherits the spirit of von Neumann and Morgenstern (1944) plays no important role in the proofs of propositions in the literature, because the strongest objection to proposals of payoffs is made by all the firm when licensee firms cannot obtain at least the status quo level of payoff, i.e., $L(0)$. In this paper, every firm can obtain $L(0)$ even when it solely breaks off the negotiations, which is guaranteed by Assumption 2 (i) for effectiveness relation. It is, however, easy to confirm that the necessary and sufficient condition for the core defined in Section 4 to be nonempty and the core itself are completely the same as the ones derived in the model of Watanabe and Muto (2008). This fact may imply that the two models are not remarkably different. More sophisticated comparisons of the patent holder's profits may thus follow in future research.

Finally, Example 2 (at the end of Section 4) implies that farsighted stable sets defined for abstract games may be closely related to those defined for games with transferable utility under some condition, whereas Example 3 (in the middle of Section 5) suggests that it be difficult to directly apply the latest result of Ray and Vohra (2019a) to abstract games. Our argument on the relationship between those solution concepts is here limited to some simple examples, but there might be a vast ocean which more researchers should have explored in detail.

[q] Kimya (2018) includes a review on various strands of the literature on farsightedness. For example, this paper assumed that all the players are farsighted, but Herings et al. (2017) studied matching when some players are myopic and the others are farsighted. Farsightedness is applied also to network formation: Dutta et al. (2005), Herings et al. (2009), and Herings et al. (2018).

A Appendix

In the main part of this paper, a patent licensing agreement is assumed to be multilateral so that not only the patent holder but also a licensee firm can solely dissolve the agreement and induce a null outcome as Assumption 2(i). Alternatively, we may consider a situation where the patent holder reaches bilateral contracts with licensee firms simultaneously. Such a situation can be formulated by an effectivity relation that allows residual coalition to stay in tact when a part of licensee firms deviate from an agreement as follows: for any $(S, x) \in X$ and nonempty $T \subsetneq S$, T can induce $(S \setminus T, x')$ from (S, x) for some x' such that $(S \setminus T, x') \in X$. To define such an alternative effectivity relation formally, we should impose some restriction on x'. If deviating coalition T of firms can select their favorable x', then it violates coalitional sovereignty as argued by Ray and Vohra (2015a). In this appendix, we consider two presumable cases named the fixed-payment model and fixed-licensee-profit model. Unfortunately, $\bar{X}(\alpha)$ defined in Section 4 may not be a farsighted stable set for some α in both models.

A.1 Fixed-payment model

We introduce an effectivity relation $\rightarrow^P$ for a situation where the patent holder and licensee firms reach agreements on payments of licensee firms. Therefore, when a part of licensee firms deviate from an outcome, residual licensee firms pay same amounts to the patent holder as the former outcome. Note that $W(s) - x_i$ is the payment of a licensee firm $i \in S$ at $(S, x) \in X$.

Assumption 3 *(i) For any $(S, x) \in X$ with $S \neq \emptyset$, $(S, x) \rightarrow^P_T (\emptyset, x^\emptyset)$ if and only if $T = \{0\}$ or S; (ii) For any $(S, x) \in X$ with $S \neq \emptyset$, $(S, x) \rightarrow^P_T (S \setminus T, x')$, where $x'_0 = (s - t)W(s - t) - \sum_{i \in S \setminus T} x'_i$, $x'_i = W(s - t) - (W(s) - x_i)$ for all $i \in S \setminus T$, and $x'_i = L(s - t)$ for all $i \in (N \setminus S) \cup T$, if and only if $\emptyset \neq T \subsetneq S$; (iii) For any $(S, x), (S', x') \in X$ with $S' \neq \emptyset$, $(S, x) \rightarrow^P_T (S', x')$ if and only if $T = \{0\} \cup S'$.*

Assumption 3(ii) is the feature of the fixed-payment model mentioned above. Assumption 3(i) requires that an agreement can be dissolved only by the patent holder or consent from all licensee firms, and (iii) is just same as Assumption 2(ii). The indirect dominance relation defined with $\rightarrow^P$ is denoted by $\succ^P$.

We show an example where $\bar{X}(\alpha)$ is not a farsighted stable set defined by $\succ^P$ for some $\alpha \in A$.

Example 4 Let $N = \{1, 2, 3\}$. The profit functions W and L are defined as follows.

s	0	1	2	3
$W(s)$	-	200	135	120
$L(s)$	100	0	0	-

Let $\alpha = 10$. Then, $B(10) = \{1\}$ and

$$\bar{X}(10) = \{(\{1\}, 90, 110, 0, 0), (\{2\}, 90, 0, 110, 0), (\{3\}, 90, 0, 0, 110)\}.$$

For each $i = 1, 2, 3$, denote $(\{i\}, x^{\{i\}}) \in \bar{X}(10)$, where $x_0^{\{i\}} = 90$, $x_i^{\{i\}} = 110$, and $x_j^{\{i\}} = 0$ for $j \neq i$. We show that $\bar{X}(10)$ is not a farsighted stable set. Consider $(N, y) = (N, 210, 110, 20, 20)$. At (N, y), 1 pays 10, 2 pays 100, and 3 pays 100. Suppose that there exists some $i = 1, 2, 3$ such that $(\{i\}, x^{\{i\}}) \succ^P (N, y)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (N, y)$, $(Q^m, z^m) = (\{i\}, x^{\{i\}})$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow^P_{R^h} (Q^h, z^h)$ and $x_j^{\{i\}} > z_j^{h-1}$ for all $j \in R^h$. Then, $R^1 = \{i\}$ since only i is made better off at $(\{i\}, x^{\{i\}})$ than (N, y). Moreover, $R^1 = \{i\} \subseteq \{2, 3\}$ by $y_0 = 210 > 90 = x_0^{\{i\}}$ and $y_1 = 110 \geq x_1^{\{j\}}$ for all $j = 1, 2, 3$. By the symmetry, we may assume that $i = 2$. Then, only $(N, y) \rightarrow^P_{\{2\}} (\{1, 3\}, 110, 125, 0, 35)$ is possible, where obviously $(\{1, 3\}, 110, 125, 0, 35) \notin \bar{X}(10)$. However, $R^2 = \emptyset$ by $x_0^{\{2\}} = 90 < 110$, $x_1^{\{2\}} = 0 < 125$, and $x_3^{\{2\}} = 0 < 35$, contradicting that R^2 is a coalition. Hence, $\bar{X}(10)$ does not satisfy external stability.

A.2 Fixed-licensee-profit model

We introduce an effectivity relation $\rightarrow^L$ for a situation where the patent holder and licensee firms reach agreements on profits of licensee firms. Therefore, when a part of licensee firms deviate from an outcome, residual licensee firms are guaranteed the same payoff as the former outcome.

Assumption 4 *(i) For any $(S, x) \in X$ with $S \neq \emptyset$, $(S, x) \rightarrow^L_T (\emptyset, x^\emptyset)$ if and only if $T = \{0\}$ or S. (ii) For any $(S, x) \in X$ with $S \neq \emptyset$, $(S, x) \rightarrow^L_T (S \setminus T, x')$, where $x'_0 = (s - t)W(s - t) - \sum_{i \in S \setminus T} x'_i$, $x'_i = x_i$ for all $i \in S \setminus T$, and $x'_i = L(s - t)$ for all $i \in (N \setminus S) \cup T$, if and only if $\emptyset \neq T \subsetneq S$. (iii) For any $(S, x), (S', x') \in X$ with $S' \neq \emptyset$, $(S, x) \rightarrow^L_T (S', x')$ if and only if $T = \{0\} \cup S'$.*

Assumption 4(ii) is the feature of the fixed-licensee-profit model mentioned above. Assumption 4(i) requires that an agreement can be dissolved only by the patent holder or consent from all licensee firms, and (iii) is just same as Assumption 2(ii). The indirect dominance relation defined with $\rightarrow^L$ is denoted by $\succ^L$.

We show an example where $\bar{X}(\alpha)$ is not a farsighted stable set defined by $\succ^L$ for some $\alpha \in A$.

Example 5 Let $N = \{1, 2, 3\}$. The profit functions W and L are defined as follows.

s	0	1	2	3
$W(s)$	-	190	160	130
$L(s)$	100	0	0	-

Let $\alpha = 10$. Then, $B(10) = \{2\}$ and

$$\bar{X}(10) = \{(\{1,2\}, 100, 110, 110, 0), (\{2,3\}, 100, 0, 110, 110), (\{1,3\}, 100, 110, 0, 110)\}.$$

We show that $\bar{X}(10)$ is not a farsighted stable set. Suppose that $\bar{X}(10)$ is a farsighted stable set. Consider $(N, y) = (N, 100, 110, 110, 70)$. By external stability of $\bar{X}(10)$, there exists some $(S^*, x^*) \in \bar{X}(10)$ such that $(S^*, x^*) \succ^L (N, y)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (N, y)$, $(Q^m, z^m) = (S^*, x^*)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow^L_{R^h} (Q^h, z^h)$ and $x^*_j > z^{h-1}_j$ for all $j \in R^h$. For all $(S, x) \in \bar{X}(10)$, $y_0 = 100 = x_0$ and $y_i = 110 \geq x_i$ for $i = 1, 2$. Thus, $R^1 = \{3\}$. Then, $(Q^1, z^1) = (\{1, 2\}, 100, 110, 110, 0) \in \bar{X}(10)$. However, $m > 1$ since $y_3 = 70 > 0 = z^1_3$. Then, $(Q^1, z^1) \rightarrow_{R^2} (Q^2, z^2) \rightarrow_{R^3} \cdots \rightarrow_{R^m} (Q^m, z^m) = (S^*, x^*)$ yield $(S^*, x^*) \succ^L (Q^1, z^1) = (\{1, 2\}, 100, 110, 110, 0)$ by the definition of an indirect dominance relation, contradicting internal stability of $\bar{X}(10)$. Hence, $\bar{X}(10)$ is not a farsighted stable set.

References

Bloch, F., van den Nouweland, A. (2017) Farsighted stability with heterogeneous expectations. mimeo.

Chander, P. (2015) A farsighted stable set for partition function games. mimeo.

Chwe, M. S.-Y. (1994) Farsighted coalitional stability. J. Econ. Theory 63, 299-325.

Diamantoudi, E. (2005) Stable cartels revisited. Econ. Theory 26, 907-921.

Diamantoudi, E., Xue, L. (2003) Farsighted stability in hedonic games. Soc. Choice Welfare 21, 39-61.

Dutta, B., Ghosal, S., Ray, D. (2005) Farsighted network formation. J. Econ. Theory 122, 143-164.

Dutta, B., Vartiainen, H. (2017) Coalition formation and history dependence. mimeo.

Dutta, B., Vohra, R. (2017) Rational expectations and farsighted stability. Theor. Econ. 12, 1191-1227.

Fan, C., Jun, B., Wolfstetter, E. (2016). Optimal bid disclosure in patent license auctions under alternative modes of competition. Int. J. Ind. Organ. 47, 1-32.

Harsanyi, J. (1974) An equilibrium-point interpretation of stable sets and a proposed alternative definition. Manage. Sci. 20, 1472-1495.

Herings, P. J.-J., Mauleon, A., Vannetelbosch, V. (2009) Farsightedly stable networks. Game. Econ. Behav. 67, 526-541.

Herings, P. J.-J., Mauleon, A., Vannetelbosch, V. (2017) Matching with myopic and farsighted players. mimeo.

Herings, P. J.-J., Mauleon, A., Vannetelbosch, V. (2018) Stability of networks under horizon-k farsightedness. Econ. Theory, forthcoming

Hirai, T. (2018) Single-payoff farsighted stable sets in strategic games with dominant punishment strategies. Int. J. Game Theory 47, 1087-1111.

Hirai, T., Watanabe, N. (2018) von Neumann-Morgenstern stable sets of a patent licensing game: the existence proof. Math. Soc. Sci. 94, 1-12.

Jordan, J. S. (2006). Pillage and property. J. Econ. Theory 13, 26-44.

Kamien, M. I., Tauman, Y. (1984) The private value of a patent: a game theoretic analysis. J. Econ. Suppl. 4, 93-118.

Kamien, M. I., Tauman, Y. (1986) Fees versus royalties and the private value of a patent. Q. J. Econ. 101, 471-491.

Kimya, M. (2018), Equilibrium coalitional behavior. mimeo.

Kishimoto, S. (2013) Stable bargaining outcomes in patent licensing: A cooperative game approach without side payments. Math. Soc. Sci. 66, 183-195.

Kishimoto, S. Watanabe, N., (2017) The kernel of a patent licensing game: The optimal number of licensees. Math. Soc. Sci. 86, 37-50.

Kishimoto, S., Watanabe, N., Muto, S. (2011) Bargaining outcomes in patent licensing: Asymptotic results in a general Cournot market. Math. Soc. Sci. 61, 114-123.

Konishi, H., Ray, D. (2003) Coalition formation as a dynamic process. J. Econ. Theory 110, 1-41.

Mauleon, A., Vannetelbosch, V., Vergote, W. (2011) von Neumann-Morgenstern farsighted stable sets in two-sided matching. Theor. Econ. 6, 499-521.

Ray, D., Vohra, R. (2014) Coalition Formation. In: Handbook of Game Theory, vol. 4, H. P. Young and S. Zamir (eds.), 239-326. North Holland, Elsevier.

Ray, D., Vohra, R. (2015a) The farsighted stable sets. Econometrica 83, 977-1011.

Ray, D., Vohra, R. (2015b) Supplement to "The farsighted stable sets" (Econometrica 83, 977-1011).
https://debrajray.com/wp-content/uploads/2017/12/RayVohraFarSightedSuppl.pdf

Ray, D., Vohra, R. (2019a) Maximality in the farsighted stable set. Econometrica, *forthcoming*.

Ray, D., Vohra, R. (2019b) Maximality in the farsighted stable set: Online Appendix.
https://www.econometricsociety.org/sites/default/files/16047_OnlineAppendix.pdf

Sen, D., Tauman, Y. (2007) General licensing schemes for a cost-reducing innovation. Game. Econ. Behav. 59, 163-186.

Tauman, Y., Watanabe, N. (2007) The Shapley value of a patent licensing game: The asymptotic equivalence to non-cooperative results. Econ. Theory 30, 135-149.

von Neumann, J., Morgenstern, O. (1944) Theory of Games and Economic Behavior, 1st Edition. Princeton University Press, Princeton, NJ, [2nd Edn. 1947], [3rd Edn. 1953].

Watanabe, N., Muto, S. (2008) Stable profit sharing in patent licensing: General bargaining outcomes. Int. J. Game Theory 37, 505-523.